Oberlin College
CSCI 383
Fall 2026
Lecture 24. Undeciability
Kozen - Section 32

Last Time: Encoding Turing machines

Consequence: Universal Turing machine.

Membership problem: MP $= \{N\#x \mid N$ encodes a Turing machine that accepts $x\}$

Halting problem: HP $= \{N\#x \mid N$ encodes a Turing machine that halts on $x\}$

Using universal TM: MP, HP are RE.

Today: Recursive?

$\Rightarrow$ We'll begin exploring the fundamental limits of computation.

A Thought Experiment

Assume that there a total TM M for MP.

Inputs: $N\#x$, where N is the encoding of a TM and $x \in \{0,1\}^*$.

Every Turing machine encoding is a string on $\{0,1\}^*$.

$\Rightarrow$ there's nothing stopping x from being an encoded TM.

Consider the following TM M^*, which takes input an encoded Turing machine N:

- Run M on $N\#N$.
- If M accepts: reject. If M rejects: accept.

Question: Total?

Question: Does M^* accept M^* or reject it?

No good answer:

- If M^* accepts M^*, then M will reject $M^*\#M^*$. But then M^* rejects M^*.
- If M^* rejects on M^*, then M will accept $M^*\#M^*$. But then M^* accepts M^*.

Hence, M cannot exist.

Diagonalization

Idea: Graphical interpretation.

TM encodings are strings.$\Rightarrow$ Can be put in lexicographic order.

Can construct a membership table. Entry in row N, column N':

- True if N accepts N'.

- False if N rejects on N'.

Draw out 5 by 5 example.

M = a total TM for looking up entries of this table.

This table contains every Turing machine.

It must contain M^*.

Write out what M^*'s row must look like.

This can't be any row!

The Halting Problem

Goal: Use diagonalization to prove HP is not recursive.

Assume we had a total TM M' for HP.

M' looks up entries in the following table:

- Rows and cols: TMs
- Entries: Does N halt on N'?

Can we come up with a TM for the flipped diagonal row?

Non-RE Languages

What's the difficulty with MP?

- Easy to determine if a TM accepts.
- Easy to determine if a TM halts and rejects.
- Problem: Determining if a TM loops.

Define $\sim$ MP $= \{M\#x \mid M \text{ rejects or loops on } x\}$

Assume $\sim$ MP was RE.

Give a total TM for MP.

- Run TMs for MP, $\sim$ MP in parallel.
- Accept if first TM accepts, reject if second TM accepts.

Conclusion: $\sim$ MP is not RE.

(Neither is $\sim$ HP, by a similar argument.)