Oberlin College
CSCI 383
Fall 2026
Lecture 19. Single-State NPDAs
Kozen - Section 25

Last few classes:

- CFGs = CFGs in CNF
- CFGs in CNF = NPDAs w/ one state, accept by empty stack
  Recall: Acceptance agnostic NPDAs: clear stack iff machine is in state f
- Acceptance Agnostic NPDAs = NPDAs

Today: AANPDAs = NPDAs with one state (accepting by empty stack)

Idea: Keep track of states on stack.

If our machine doesn't need the stack, this is easy.

More general issue: If we need to use the stack, we'll need to clear symbols at some point.

If we clear the top symbol, then we've lost track of our current state.

Stack Labelings

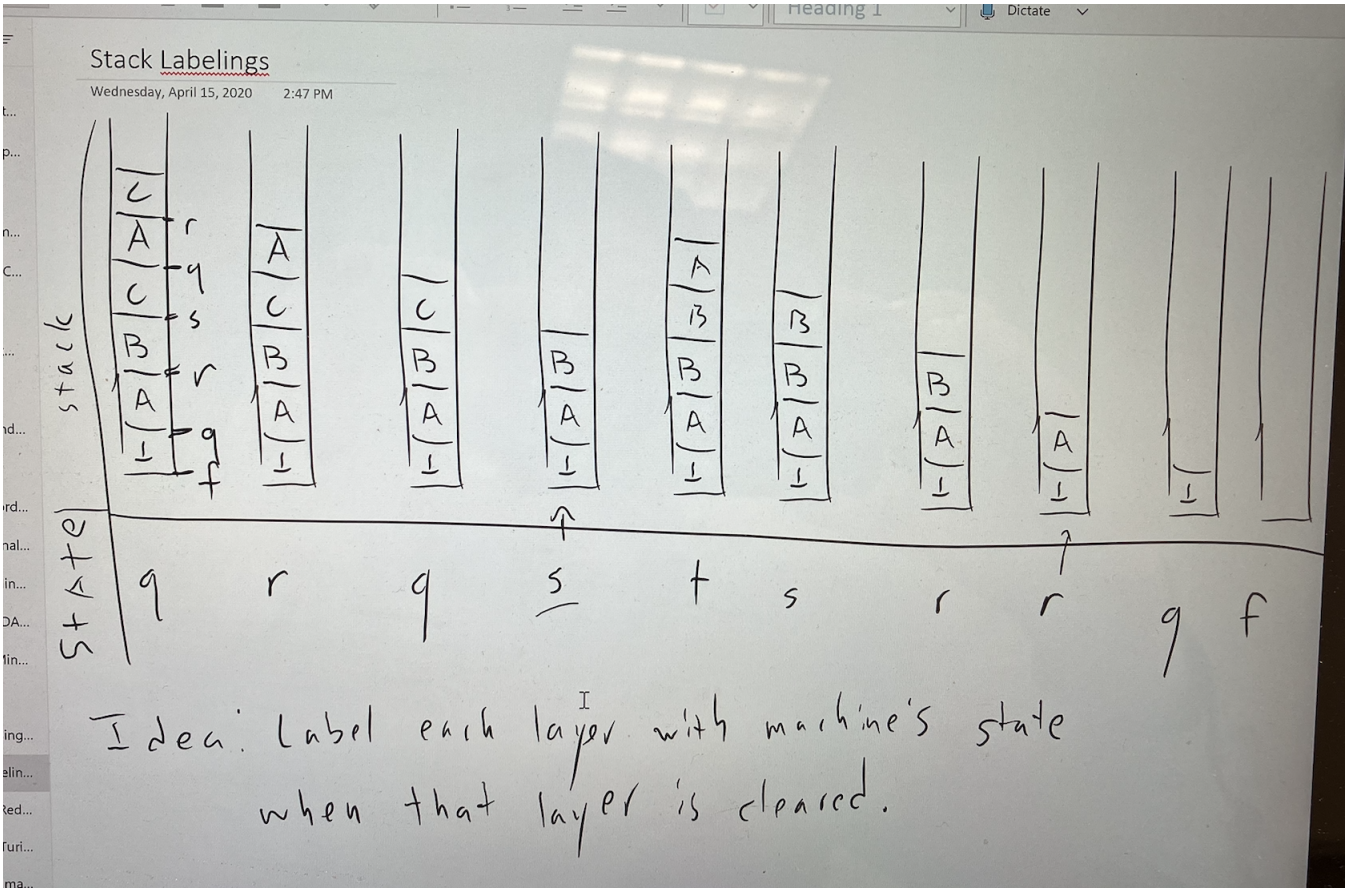
Idea: Record state whenever stack size decreases.

Consider a run of some NPDA.

A layer is cleared if there are no stack symbols above that layer. (They could get added back later.)

Goal: Label initial stack contents with the state when each layer is cleared.

Work out on example.



Single-State NPDAs

Let $M = (Q, \Sigma, \Gamma, \delta, s, \bot, \{f\})$ be an acceptance-agnostic NPDA with final state f.

Idea: Two things:

- Track the current state.
- Guess a stack labeling.

New NPDA $M' = (\{*\}, \Sigma, \Gamma', \delta', *, \quad)$ accepting by empty stack.

Stack symbols: $\langle pAq \rangle \in Q \times \Gamma \times Q$.

- p - current state
- A - stack symbol
- q - guess: what's the state when this layer gets cleared?

Transitions δ': Let $((p, c, A), (r, B_1 \dots B_k))$ be a transition in δ.

New transitions: $((*, c, \langle pAq_k \rangle), (*, \langle rB_1q_1 \rangle \langle q_1B_2q_2 \rangle \dots \langle q_{k-1}B_kq_k \rangle))$

(For all $q_1, \dots, q_k \in Q$.)

Thing to check: Top symbol tracks current state.

The rest of it: A promised stack labeling.

What if we push no stack symbols? (What should it be, based the "guessed stack labeling" story?)

- $((p, c, A), (r, \epsilon))$ becomes
- $((*, c, \langle pAr\rangle), (*, \epsilon))$

What should the initial stack symbol be in M'?

- Top symbol tracks current state.
- What's the initial stack symbol in M?
- Where do we need to be when we clear the stack totally?

$\langle s \bot f\rangle$

Proving Correctness

Key intuition: $((p, c, A), (r, \epsilon))$ becomes $((*, c, \langle pAr\rangle), (*, \epsilon))$

Three stages to consider:

- Initial guess: $(*, x, \langle pB_1q_1\rangle\langle q_1B_2q_2\rangle \dots \langle q_{k-1}B_kr\rangle)$
- After performing some transitions: $(*, y, \langle qC_1p_1\rangle\langle p_1C_2p_2\rangle \dots \langle p_{m-1}C_mq_1\rangle\langle q_1B_2q_2\rangle \dots \langle q_{k-1}B_kr\rangle)$
- Finally clearing the next layer: $(*, y, \langle p_{m-1}C_mq_1\rangle\langle q_1B_2q_2\rangle \dots \langle q_{k-1}B_kr\rangle)$

At all times, the top of the stack tracks the state.

We can make the last transition to clear the next layer if and only if our guess of the state was correct!

More General Idea: Can clear $B_1 \dots B_k$ if and only if you can guess a stack labeling that works for M'.

Thm: $(p, x, B_1 \dots B_k) \xrightarrow[M]{n} (r, \epsilon, \epsilon)$ if and only if there exists there exist $q_1, \dots, q_{k-1}$ such that

$$(*, x, \langle pB_1q_1\rangle\langle q_1B_2q_2\rangle \dots \langle q_{k-1}B_kr\rangle) \xrightarrow[M']{n} (*, \epsilon, \epsilon)$$

Proof: (Induction on n).

($\Rightarrow$) Assume $(p, x, B_1 \dots B_k) \xrightarrow[M]{n+1} (r, \epsilon, \epsilon)$.

Let $((p, c, B_1), (q, C_1C_2 \dots C_m))$ be the first transition applied.

Then we know:

- $x = cy$ for some y
- $(p, x, B_1 \dots B_k) \xrightarrow[M]{1} (q, y, C_1 \dots C_mB_1 \dots B_k) \xrightarrow[M]{n} (r, \epsilon, \epsilon)$

By induction, we have: that there exist $p_1, \dots, p_{m-1}$ and $q_1, \dots, q_{k-1}$ such that

$$(*, y, \langle qC_1p_1\rangle\langle p_1C_2p_2\rangle \dots \langle p_{m-1}C_mq_1\rangle\langle q_1B_2q_2\rangle \dots \langle q_{k-1}B_kr\rangle) \xrightarrow[M']{n} (*, \epsilon, \epsilon)$$

But we also have that $((*, c, \langle qB_1q_1\rangle), (*, \langle qC_1p_1\rangle\langle p_1C_2p_2\rangle \dots \langle p_{m-1}C_mq_1\rangle)$ is a production in M'.

Hence $(*, x, \langle pB_1q_1\rangle\langle q_1B_2q_2\rangle \dots \langle q_{k-1}B_kr\rangle) \xrightarrow[M']{n} (*, \epsilon, \epsilon)$.

Reverse direction is the same argument in reverse.