Oberlin College
CSCI 383
Fall 2026
Lecture 18. PDAs vs. CFGs
Kozen - Section 24

Warmup. Consider the following NPDA.

- $Q = \{q\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{\bot, A, B\}$
- δ :
  - $((q, \epsilon, \bot), (q, A \bot B))$
  - $((q, a, A), (q, \epsilon))$
  - $((q, b, B), (q, \epsilon))$
- $s = q$
- $\bot$ is $\bot$
- acceptance by empty stack

Find an accepting path for ab, $aabb$

Draw CFG, work through the derivation.

Observation: NPDA just simulates the CNF CFG for $\{a^n b^n\}$.

Def: Given a CFG G and a string $x \in L(G)$, a derivation of x in G is a *leftmost derivation* if the production applied at each step is to the leftmost nonterminal in the sentential form.

Accepting paths corresponded to leftmost derivations.

CFGs to NPDAs

Example: Convert the following grammar to an NPDA

$$\begin{aligned} &\bot \to \bot\bot \,|\, AC \,|\, AD \,|\, \epsilon \\ &C \to \bot B \\ &D \to \bot A \\ &A \to a \\ &B \to b \end{aligned}$$

Example: Convert the following grammar to an NPDA:

$$\bot \to 0\bot 0 \,|\, 0\bot 1 \,|\, 1\bot 0 \,|\, 1\bot 1 \,|\, 0$$

This second example doesn't work! Nonterminals at the right of productions.

Solution: CNF.

In fact, there's nothing stopping us from doing this in general.

Procedure: $G = (\Sigma, N, P, S)$, construct NPDA as follows

- $Q = \{q\}$
- Σ stays the same
- $\Gamma = N$
- δ:
  - For every production of form $A \to BC$, add production $((q, \epsilon, A), (q, BC))$.
  - For every production of form $A \to a$, add production $((q, a, A), (q, \epsilon))$.
- $s = q$
- $\perp = S$
- acceptance by empty stack

NPDAs to CFGs

Observation: Above construction only yields NDPAs with a single state.

Given NPDA with single state, can we reverse this procedure?

Given NPDA $M = (Q, \Sigma, \Gamma, \delta, q, \perp)$ with one state, accepting by empty stack:

- Σ stays the same
- $N = \Gamma$
- $S = \perp$
- Productions: For any transition of form $((q, c, A), (q, \gamma))$, add production $A \to c\gamma$

Correctness. Given a valid run of the NPDA, can construct a derivation in the CFG by applying productions corresponding to the transitions.

$\Rightarrow$ inductive proof of one direction.

What about the other direction? Can we take an arbitrary derivation from CFG?

How do we simulate steps of the form $ABCD \xrightarrow[G]{1} ABC\gamma$?

Solution: Leftmost derivations.

Def: Given a CFG G and a string $x \in L(G)$, a derivation of x in G is a *leftmost derivation* if the production applied at each step is to the leftmost nonterminal in the sentential form.

Fact: Every $x \in L(G)$ has a leftmost derivation.

Theorem: Given an NPDA M and CFG G related by either the forward or reverse construction above, $(q, x, S) \xrightarrow[M]{n} (q, \epsilon, \epsilon)$ if and only if $S \xrightarrow[G]{n} x$ by leftmost derivation.

Proof: Induction on number of steps of derivation.