

Before break: Limits of CFLs.

- $\{a^n b^n \mid n \geq 0\}$
- $\{a^{3^n} \mid n \geq 0\}$
- $\{a^n b^n a^n \mid n \geq 0\}$
- $\{x \in \{(,)\}^* \mid x \text{ balanced}\}$

Very high-level intuition: counting, but not too much.

Today: A model of computation that accepts CFLs.

Nondeterministic Pushdown Automata (NPDAs)

Example. Transitions:

- $((q_0, a, \perp), (q_0, A\perp))$
- $((q_0, a, A), (q_0, AA))$
- $((q_0, \epsilon, A), (q_1, A))$
- $((q_1, b, A), (q_1, \epsilon))$
- $((q_1, \epsilon, \perp), (q_1, \epsilon))$

Does the NPDA accept these?

- ab
- abb
- b
- $aabb$
- ϵ

What's the language? $\{a^n b^n \mid n \geq 1\}$.

Formal Definition

An NPDA is a 6-tuple $M = (Q, \Sigma, \Gamma, \delta, s, \perp)$

- Q - finite state space.
- Σ - input alphabet (or “tape alphabet”)
- Γ - stack alphabet
- $\delta \subseteq (Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma) \times (Q \times \Gamma^*)$ - transitions.

Interpretation of $((p, c, A), (q, \gamma))$: Reading character c and popping A off the stack in state p , transition to state q and push γ onto the stack.

- s - start state
- $\perp \in \Gamma$ - empty stack symbol.

Define all these things for the example above.

Exercise. Below are some, but not all, of the transitions for an NPDA for $\{x \in \{a, b\}^* \mid \#_a(x) = \#_b(x)\}$.

- $((q, a, \perp), (q, A\perp))$
- $((q, a, A), (q, AA))$
- $((q, b, A), (q, \epsilon))$
- $((q, \epsilon, \perp), (q, \epsilon))$

To see what's going on, try this on $aaabbb$, $aaaabb$, ϵ and $aabbb$.

How do we get it to accept $bbbaaa$ without accepting some of the faulty strings above? How about $ababab$?

Describe in English what is going on with this NPDA.

Configurations and Acceptance

Idea: Track current status of computation.

What information is relevant?

- Current state.
- Characters left on tape.
- Characters still on stack.

Def: A *configuration* is a triple $(q, x, \gamma) \in Q \times \Sigma^* \times \Gamma^*$, where:

- q is the current state.
- x is the characters left on the tape.
- γ is the characters left on the stack.

The initial configuration for an NPDA reading x is $(s, x, \perp)$.

Given two configurations $(p, cx, A\gamma)$ and $(q, x, \beta\gamma)$ of NPDA M , we write $(p, cx, A\gamma) \xrightarrow[M]{1} (q, x, \beta\gamma)$ if $((p, c, A), (q, \beta)) \in \delta$.

We can extend the above multiple transitions in the following way. Given configurations C and D :

- $C \xrightarrow[M]{0} D$ if $C = D$
- $C \xrightarrow[M]{n+1} D$ for $n \geq 1$ if $C \xrightarrow[M]{n} E$ and $E \xrightarrow[D]{1}$ for some configuration E
- $C \xrightarrow[M]{*} D$ if $C \xrightarrow[M]{n} D$ for some $n \geq 0$.

NPDA M *accepts* $x \in \Sigma^*$ by empty stack if $(s, x, \perp) \xrightarrow[M]{*} (q, \epsilon, \epsilon)$ for some $q \in Q$.

Can prove NPDAs are correct by induction, just like we did with DFAs/NFAs.

Example: In Exercise above, $(q, xy, \perp) \xrightarrow[M]{*} (q, y, \gamma\perp)$ if and only if one of the following holds:

- $\#_a(x) - \#_b(x) = j > 0$ and $\gamma = A^j\perp$

- $\#_b(x) - \#_a(x) = j > 0$ and $\gamma = B^j \perp$
- $\#_a(x) - \#_b(x) = 0$ and $\gamma = \perp$

Alternative formulation: acceptance by final state.

Add to definition: $f \in Q$ - final state.

Def: An NPDA $M = (Q, \Sigma, \Gamma, \delta, s, \perp, f)$ (with f necessary!) accepts $x \in \Sigma$ by final state if $(s, x, \perp) \xrightarrow{*}_M (f, \epsilon, \gamma)$, for $\gamma \in \Gamma^*$.

Theorem: For any NPDA M , accepting by empty stack, with $L(M) = A$, there exists another NDPA M' , accepting by final state, with $L(M') = A$, and vice versa.

Proof is a fun exercise.

What would really be nice: if it doesn't matter which notion of acceptance you use.

Def. An NPDA $M = (Q, \Sigma, \Gamma, \delta, s, \perp, F)$ is *acceptance-agnostic* if for every string x ,

- If $(s, x, \perp) \xrightarrow{*}_M (q, \epsilon, \epsilon)$, then $q = f$.
- $(s, x, \perp) \xrightarrow{*}_M (f, \epsilon, \beta)$, then $(s, x, \perp) \xrightarrow{*}_M (f, \epsilon, \epsilon)$

Theorem: For any NPDA M , accepting by empty stack or final state, with $L(M) = A$, there exists another acceptance-agnostic NDPA M' with one final state, such that with $L(M') = A$.

Proof: Kozen, Section E.

Exercise: Change our example to be acceptance-agnostic.