Example proof

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January 22, 2014

We want to prove the following theorem.

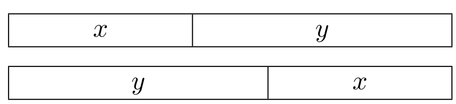
Theorem. *Let Σ be a finite alphabet and let $w \in \Sigma^*$ be a word of length at least 2. If there exist nonempty words $x, y \in \Sigma^+$ such that $w = xy = yx$, then there exists a nonempty word $z \in \Sigma^+$ and positive integers a and b such that $x = z^a$ and $y = z^b$.*

Proof. We will prove the theorem by induction on the length of w. The base case is $|w| = 2$. This case is trivial since it must be the case that $|x| = |y| = 1$. Therefore, $x = y$ so $z = x$ and $a = b = 1$.

For the inductive step, assume that the theorem holds for all words of length less than m. Let w be a word of length m such that $w = xy = yx$ for nonempty x and y. Write $x = x_1 x_2 \cdots x_k$ and $y = y_1 y_2 \cdots y_n$. We can assume, without loss of generality, that $k \le n$ (otherwise swap x and y).

If $k = n$, then since $xy = yx$, we have $x_1 = y_1$, $x_2 = y_2$, $\ldots$, $x_k = y_k$. Therefore, $x = y$ and we can set $z = x$ and $a = b = 1$.

Otherwise, $k < n$ and we have a situation that looks like this.



From the picture, it's clear that y starts with x and ends with the first $n - k$ letters in y. Formally, $y = xy_1 y_2 \cdots y_{n-k}$. Similarly, y ends with x so $y = y_1 y_2 \cdots y_{n-k} x$. Define $y' = y_1 y_2 \cdots y_{n-k}$. Thus,

$$y = xy' = y'x$$

and we can apply the inductive hypothesis since $|y| = n < |w| = m$. In particular, there is a word z and integers a, c such that $x = z^a$ and $y' = z^c$. This gives

$$\begin{aligned} x &= z^a, \\ y &= xy' = z^a z^c = z^{a+c}. \end{aligned}$$

Setting $b = a + c$ proves the theorem. ∎