

CS 301

Lecture 19 – Diagonalization and undecidable languages

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Sizes of sets

Two sets X and Y have the same **size** if there is a bijection between them, $f : X \rightarrow Y$

What's a bijection?

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What's a bijection?

Recall $f : X \rightarrow Y$ is a bijection if

- 1 for all $a, b \in X$, $f(a) = f(b)$ implies $a = b$ (**injective**)
- 2 for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$ (**surjective**)

Example

The natural numbers and the integers have the same size

$$f : \mathbb{Z} \rightarrow \mathbb{N}$$
$$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x - 1 & \text{if } x < 0 \end{cases}$$

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$$\begin{array}{l} \vdots \\ -2 \mapsto 3 \\ -1 \mapsto 1 \\ 0 \mapsto 0 \\ 1 \mapsto 2 \\ 2 \mapsto 4 \\ \vdots \end{array}$$

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The fundamental theorem of arithmetic tells us that every positive integer can be expressed uniquely as a product of prime powers

$$p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots$$

where p_i are the primes in order (2, 3, 5, 7, etc.) and $n_i \in \mathbb{N}$ and finitely many n_i are nonzero

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where p_i are the primes in order (2, 3, 5, 7, etc.) and $n_i \in \mathbb{N}$ and finitely many n_i are nonzero

Similarly, every positive rational number can be expressed uniquely as a product of prime powers

$$p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots$$

where p_i are the primes in order and $n_i \in \mathbb{Z}$ and finitely many n_i are nonzero

Example continued

Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ be our bijection from before

Define $g : \mathbb{Q}^+ \rightarrow \mathbb{Z}^+$ by

$$g(p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots) = p_1^{f(n_1)} p_2^{f(n_2)} p_3^{f(n_3)} \cdots$$

Note that we're mapping the integer exponents to natural number exponents and the (infinitely many) 0 exponents remain 0 because $f(0) = 0$

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Finally, let's define our bijection $h : \mathbb{Q} \rightarrow \mathbb{Z}$

$$h(x) = \begin{cases} g(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -g(-x) & \text{if } x < 0 \end{cases}$$

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And just for fun, $f \circ h : \mathbb{Q} \rightarrow \mathbb{N}$ is a bijection

Countable

A set X is **countable** if it is finite or it has the same size as \mathbb{N}

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Subsets of countable sets are countable (intuitively true but a hassle to prove without some additional math or an alternative, but equivalent definition of countability)

Each language is a countable set

Given an alphabet Σ , the language Σ^* is countably infinite. How do we show this?

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List the strings in lexicographic order to construct the mapping

E.g., $f : \{0, 1\}^* \rightarrow \mathbb{N}$ given by

$$\varepsilon \mapsto 0$$

$$0 \mapsto 1$$

$$1 \mapsto 2$$

$$00 \mapsto 3$$

$$01 \mapsto 4$$

$$10 \mapsto 5$$

$$11 \mapsto 6$$

$$000 \mapsto 7$$

\vdots

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Given an alphabet Σ , the language Σ^* is countably infinite. How do we show this?

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Every language $L \subseteq \Sigma^*$ is thus countable

Diagonalization: infinite sequences over $\{0, 1\}$

Theorem

The set S of all infinite sequences over $\{0, 1\}$ is uncountable

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Proof.

Assume S is countable so there's a bijection $f : \mathbb{N} \rightarrow S$

We can construct a new infinite sequence $\mathbf{b} = b_0, b_1, \dots$
that differs from every sequence in S .

n	$f(n)$
0	0 0 1 0 1 ...
1	1 0 0 0 1 ...
2	0 1 1 0 0 ...
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\vdots	\vdots

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In particular, b_i will differ from $f(i)$ in position i

$$b_i = \begin{cases} 0 & \text{if the } i\text{th element of } f(i) \text{ is } 1 \\ 1 & \text{if the } i\text{th element of } f(i) \text{ is } 0 \end{cases}$$

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$\mathbf{b} = 1100\dots$

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Now $\mathbf{b} \in S$ but for all i , $f(i) \neq \mathbf{b}$ which is a contradiction so S must not be countable \square

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There are a countable number of Turing machines

Consider any fixed binary representation of a TM

E.g., given

$$Q = \{1, 2, \dots, k\}$$

$$\Sigma = \{1, 2, \dots, m\}$$

$$\Gamma = \{1, 2, \dots, n\}$$

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{1, 2\}$$

where 1 = L and 2 = R

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$$

here's one possible representation

$$\langle \delta(q, a) \rangle = 0^r 10^b 10^d$$

where $\delta(q, a) = (r, b, d)$

$$\langle \delta \rangle = \langle \delta(1, 1) \rangle 11 \langle \delta(1, 2) \rangle 11 \dots 11 \langle \delta(k, n) \rangle$$

$$\langle M \rangle = 0^k 111 0^m 111 0^n 111 \langle \delta \rangle 111 0^{q_{\text{accept}}} 111 0^{q_{\text{reject}}}$$

Thus $\langle M \rangle$ is an element of $\{0, 1\}^*$

There are a countable number of Turing machines continued

For simplicity, for all $x \in \{0, 1\}^*$ such that x is not a valid encoding of a TM, define x to be a TM with $q_0 = q_{\text{reject}}$

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Now every binary string is a valid encoding of a TM, i.e.,

$$\{0, 1\}^* = \{\langle M \rangle \mid \langle M \rangle \text{ is a TM}\}$$

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For simplicity, for all $x \in \{0, 1\}^*$ such that x is not a valid encoding of a TM, define x to be a TM with $q_0 = q_{\text{reject}}$

Now every binary string is a valid encoding of a TM, i.e.,

$$\{0, 1\}^* = \{\langle M \rangle \mid \langle M \rangle \text{ is a TM}\}$$

Since $\{0, 1\}^*$ is countable, there are a countable number of Turing machines

There are an uncountable number of languages

Theorem

For every alphabet Σ , the set of all languages over Σ is uncountable

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Proof.

We proved that Σ^* is countably infinite; let $f : \mathbb{N} \rightarrow \Sigma^*$ be a bijection

For each language L over Σ , define an infinite sequence $\mathbf{b} = b_0, b_1, \dots$ over $\{0, 1\}$ where

$$b_i = \begin{cases} 0 & \text{if } f(i) \notin L \\ 1 & \text{if } f(i) \in L \end{cases}$$

\mathbf{b} is called the **characteristic sequence of L**

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\mathbf{b} is called the **characteristic sequence of L**

Each characteristic sequence defines a language and each language has a unique characteristic sequence

We proved that there are uncountably many infinite binary sequences so there are uncountably many languages over Σ

A simple corollary

There are (uncountably many) languages that are not Turing-recognizable (and thus not decidable)

An explicit undecidable language

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Assume that D is a TM that decides DIAG

Is $\langle D \rangle \in \text{DIAG}$?

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Two options

- If $\langle D \rangle \in \text{DIAG}$, then since D decides DIAG , D must accept $\langle D \rangle$ but then by definition of DIAG , $\langle D \rangle \notin \text{DIAG}$

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Either option leads to a contradiction so DIAG must not be decidable



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Either option leads to a contradiction so DIAG must not be decidable □

Replacing “reject” with “does not accept” in the proof shows that DIAG is not only not decidable, it’s not even Turing-recognizable!



Acceptance problem for TMs

Theorem

The language $A_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } w \in L(M)\}$ is undecidable

How should we approach problems like this?

Proving that a language is not decidable

To prove that a language A is undecidable,

- 1 Assume that A is decidable and let R be a TM that decides A
- 2 Select an undecidable language B
- 3 Construct a new TM D that decides B and that uses R as a subroutine
- 4 Since B is undecidable but D is a decider, this is a contradiction and our assumption in step 1 must be wrong so A is undecidable

Steps 2 and 3 are the hard steps that require some cleverness

Proof

Proof that A_{TM} is undecidable.

Assume that A_{TM} is decidable with decider R .

Let's build a TM D that decides DIAG.

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Let's build a TM D that decides DIAG.

$D =$ "On input $\langle M \rangle$,

- 1 Run R on $\langle M, \langle M \rangle \rangle$
- 2 If R accepts, *reject*; otherwise *accept*."

We need to show that $L(D) = \text{DIAG}$ and that D is a decider.

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- 2 If R accepts, *reject*; otherwise *accept*."

We need to show that $L(D) = \text{DIAG}$ and that D is a decider.

By assumption, R is a decider so it halts on $\langle M, \langle M \rangle \rangle$ and thus D halts on all input so it is a decider

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If $\langle M \rangle \in \text{DIAG}$, then $\langle M \rangle \notin L(M)$ so R rejects and D accepts so $\langle M \rangle \in L(D)$.

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We need to show that $L(D) = \text{DIAG}$ and that D is a decider.

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Proof that A_{TM} is undecidable.

Assume that A_{TM} is decidable with decider R .

Let's build a TM D that decides DIAG.

$D =$ "On input $\langle M \rangle$,

- 1 Run R on $\langle M, \langle M \rangle \rangle$
- 2 If R accepts, *reject*; otherwise *accept*."

We need to show that $L(D) = \text{DIAG}$ and that D is a decider.

By assumption, R is a decider so it halts on $\langle M, \langle M \rangle \rangle$ and thus D halts on all input so it is a decider

If $\langle M \rangle \in \text{DIAG}$, then $\langle M \rangle \notin L(M)$ so R rejects and D accepts so $\langle M \rangle \in L(D)$.

If $\langle M \rangle \notin \text{DIAG}$, then $\langle M \rangle \in L(M)$ so R accepts and D rejects so $\langle M \rangle \notin L(D)$.

Thus D decides DIAG. This is a contradiction so A_{TM} must not be decidable. □

Halting problem for TMs

Theorem

The language $\text{HALT}_{\text{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts when run on } w\}$ is undecidable

Assume that HALT_{TM} is decided by TM H . How do we use H to construct a decider D for A_{TM} ?

Proof

Proof.

Assume H is a decider for HALT_{TM} and build a decider D for A_{TM} .

$D =$ "On input $\langle M, w \rangle$,

- 1 Run H on $\langle M, w \rangle$ and if H rejects, *reject*.
- 2 Run M on w and if M accepts, *accept*; otherwise *reject*."

D is a decider because if M loops on w , then H and D will reject. Otherwise, M will halt on w so D will halt.

If $w \in L(M)$, then M halts on w so H will accept and then D will accept.

If $w \notin L(M)$, then there are two options. If M loops on w , then H and thus D will reject. If M rejects w , then H will accept but D will reject. \square

Co-Turing-recognizable (CoRE)

A language L is **co-Turing-recognizable** (coRE) if \bar{L} is Turing-recognizable (RE)

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A language L is **co-Turing-recognizable** (coRE) if \bar{L} is Turing-recognizable (RE)

Theorem

A language L is decidable $\iff L$ is RE and L is coRE

To prove this, we need to prove three things

- 1 If L is decidable, then L is RE
- 2 If L is decidable, then L is coRE
- 3 If L is RE and coRE, then L is decidable

Parts 1 and 2 together show the \implies direction and part 3 shows the \impliedby direction

Proof

Proof.

\implies :

If L is decidable, then there is some decider M such that $L(M) = L$. Thus L is RE.

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If L is decidable, then there is some decider M such that $L(M) = L$. Thus L is RE.

By swapping the accept and reject states of M , we get a new decider M' that decides \overline{L} . Thus L is coRE.

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\impliedby :

If L is RE, then there is some TM M_1 that recognizes it

If L is coRE, then there is some TM M_2 that recognizes \overline{L}

Build $M =$ "On input w ,

- 1 Run M_1 and M_2 on w simultaneously (e.g., with 2 tapes)
- 2 If M_1 accepts, *accept*. If M_2 accepts, *reject*."

One of M_1 or M_2 **must** accept, so M will halt on any input and thus decides L . □



A_{TM} is RE but not coRE

Theorem

A_{TM} is RE but not coRE

Proof.

Since A_{TM} is not decidable, if we show that it is RE, then it **can't** be coRE because then it would be decidable.

We can build R to recognize A_{TM} as follows.

$R =$ "On input $\langle M, w \rangle$,

- 1 Run M on w .
- 2 If M accepts, *accept*; if M rejects, *reject*."

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We can build R to recognize A_{TM} as follows.

$R =$ "On input $\langle M, w \rangle$,

- 1 Run M on w .
- 2 If M accepts, *accept*; if M rejects, *reject*."

Note that if M loops on w , then R will loop, but this is okay because R just needs to recognize A_{TM} , not decide it

Proof continued

There are three cases

- 1 $\langle M, w \rangle \in A_{\text{TM}}$. M will accept w so R will accept.

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- 1 $\langle M, w \rangle \in A_{\text{TM}}$. M will accept w so R will accept.
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Proof continued

There are three cases

- ① $\langle M, w \rangle \in A_{\text{TM}}$. M will accept w so R will accept.
- ② $\langle M, w \rangle \notin A_{\text{TM}}$. M will either loop on w or reject and R will do the same.
- ③ The input isn't a valid encoding of $\langle M, w \rangle$. R will reject before step 1.

Proof continued

There are three cases

- 1 $\langle M, w \rangle \in A_{\text{TM}}$. M will accept w so R will accept.
- 2 $\langle M, w \rangle \notin A_{\text{TM}}$. M will either loop on w or reject and R will do the same.
- 3 The input isn't a valid encoding of $\langle M, w \rangle$. R will reject before step 1.

Thus $L(R) = A_{\text{TM}}$ so A_{TM} is RE. □

Emptiness problem for TMs

Theorem

The language $E_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$ is coRE.

To prove this, we need only give a TM that recognizes $\overline{E_{TM}}$

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Proof.

Let $R =$ "On input w ,

- 1 If $w \neq \langle M \rangle$ for some TM M , *accept*.
- 2 For $n = 0$ up to ∞
- 3 For each string $w \in \Sigma^*$ of length at most n
- 4 Simulate M on w for at most n steps.
- 5 If M accepts w , *accept*."

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If $L(M) \neq \emptyset$, then there is some w that M will accept so R will accept $\langle M \rangle$.

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- 4 Simulate M on w for at most n steps.
- 5 If M accepts w , *accept*."

If $L(M) \neq \emptyset$, then there is some w that M will accept so R will accept $\langle M \rangle$.

If $L(M) = \emptyset$, then M will never accept so R will loop on $\langle M \rangle$.

Thus $L(R) = \overline{E_{TM}}$ so E_{TM} is coRE.



Emptiness problem for TMs is undecidable

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Corollary

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The language E_{TM} is not RE.

Proof of the corollary.

Since E_{TM} is coRE, if it were RE, then it would be decidable, contradicting the theorem. □

Proof idea for showing E_{TM} is undecidable

- Assume E decides E_{TM}
- Build a decider for A_{TM} using E
- Along the way, we're going to construct an entirely new TM M_w and we're going to run E on $\langle M_w \rangle$

We'll use the idea of constructing new TMs in a bunch of different proofs

Proof

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Assume that E decides E_{TM} . Build D to decide A_{TM} .

$D =$ “On input $\langle M, w \rangle$,

- ① Construct a new TM $M_w =$ ‘On any input x ,
 - ① Replace x on the tape with w and run M on w .
 - ② If M accepts, *accept*; if M rejects, *reject*.’
- ② Run E on $\langle M_w \rangle$.
- ③ If E accepts, *reject*; otherwise *accept*.”

Note that M_w is never run. It is only constructed so that $\langle M_w \rangle$ can be given as input to decider E .

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If $w \in L(M)$, then $L(M_w) = \Sigma^* \neq \emptyset$ so E rejects and D accepts.

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If $w \in L(M)$, then $L(M_w) = \Sigma^* \neq \emptyset$ so E rejects and D accepts.

If $w \notin L(M)$, then $L(M_w) = \emptyset$ so E accepts and D rejects. Thus $L(D) = E_{\text{TM}}$.

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Note that M_w is never run. It is only constructed so that $\langle M_w \rangle$ can be given as input to decider E .

If $w \in L(M)$, then $L(M_w) = \Sigma^* \neq \emptyset$ so E rejects and D accepts.

If $w \notin L(M)$, then $L(M_w) = \emptyset$ so E accepts and D rejects. Thus $L(D) = E_{\text{TM}}$.

Constructing M_w can't loop and E is a decider so D is a decider.

