

# CS 301

## Lecture 15 – Turing machines

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# Machine models of computation

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- DFA: Deterministic, finite set of states, no additional memory
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- PDA: Nondeterministic, finite set of states, infinite stack
- Turing machine (TM): Deterministic, finite set of states, one-way infinite, read-write “tape”



# Single step

One step of computation

- ① Read the symbol under the tape head
- ② Based on the current state and the symbol read,
  - ① Write a symbol to the tape (it can be the same symbol just read)
  - ② Transition to a new state
  - ③ Move the tape head left or right (in some formulations, we allow the head to stay in place)

If the tape head is at the left end of the tape and the TM tries to move left, the tape head stays in place

# TM Computation

TM computation

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- ④ Perform one step of computation
- ⑤ Goto step 2

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## TM computation

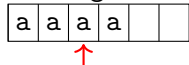
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Three possible outcomes: accept, reject, loop (run forever)



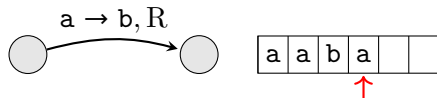
# TM transitions

Starting with the tape



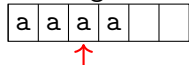
Transitions:

- Replace a with b and move right



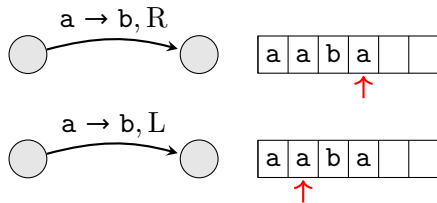
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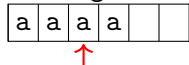
Transitions:

- Replace a with b and move right
- Replace a with b and move left



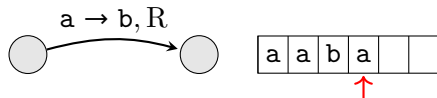
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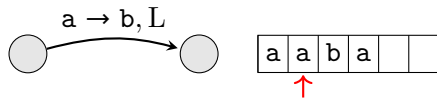


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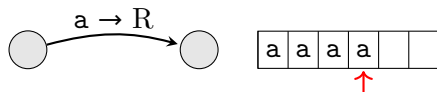


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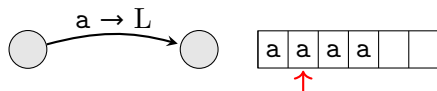


When the tape cell doesn't change, we use a shorthand

- On a, move right



- On a, move left

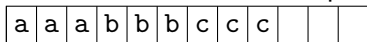


## Example

Consider building a TM to recognize the language  $A = \{a^n b^n c^n \mid n \geq 0\}$

Let's start by thinking about the algorithm we would like the TM to perform

- 1 The TM starts with the input (say  $w = aaabbbccc$  is our input) on the tape

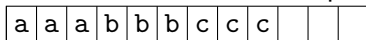


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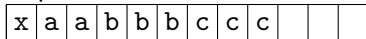
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---	---	---	---	---	---	---	---	---	--	--	--



- 2 Replace the a with an x and then scan right to a b

x	a	a	b	b	b	c	c	c			
---	---	---	---	---	---	---	---	---	--	--	--



- 3 Replace the b with a y and scan right to a c

x	a	a	y	b	b	c	c	c			
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- 5 Move right

x	a	a	y	b	b	z	c	c			
---	---	---	---	---	---	---	---	---	--	--	--



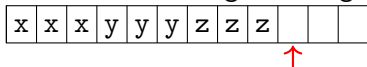


## Example

- 6 If the cell under the tape head is a, goto step 1

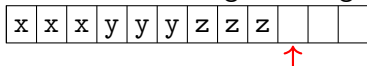
## Example

- 6 If the cell under the tape head is a, goto step 1
- 7 Otherwise, scan right through all x, y, and z



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- ⑥ If the cell under the tape head is a, goto step 1
- ⑦ Otherwise, scan right through all x, y, and z

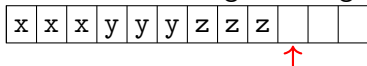


- ⑧ If the cell under the tape head is  $\square$ , accept

If at any point in the process the TM encounters a symbol it isn't expecting, reject

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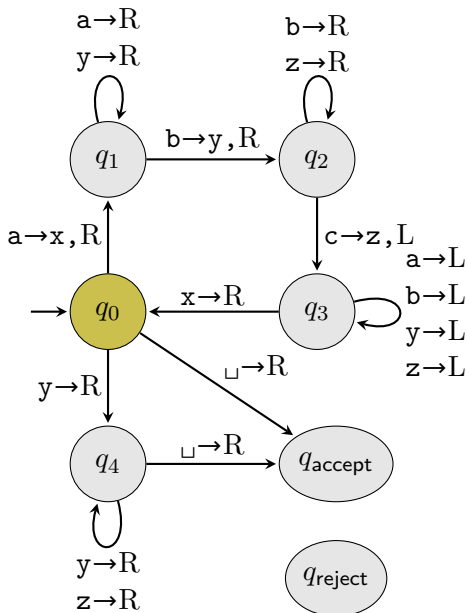
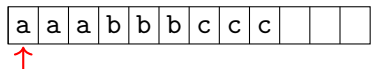
What should the TM do if the very first symbol is  $\square$ ?

It should accept because the string  $\varepsilon \in A$

# Example

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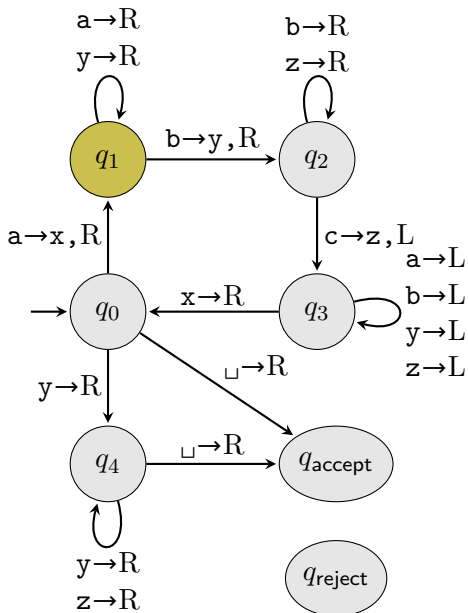
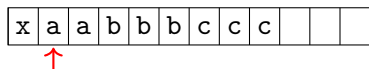
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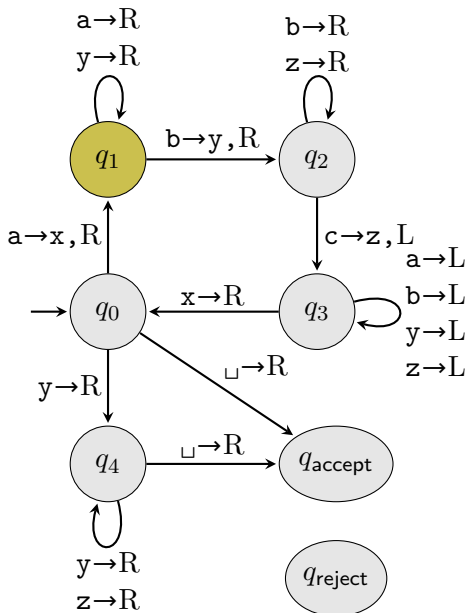
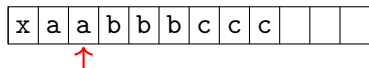
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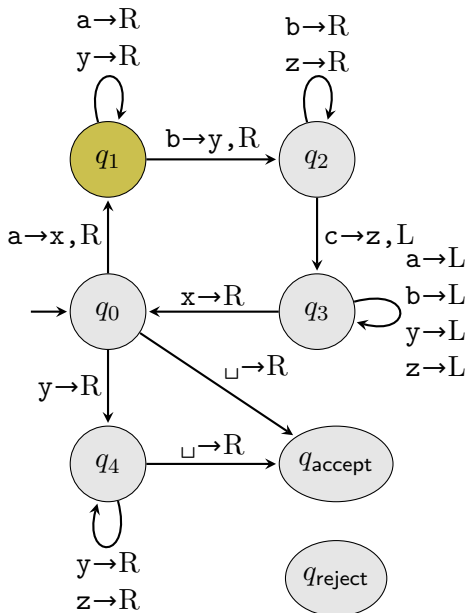
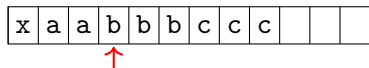
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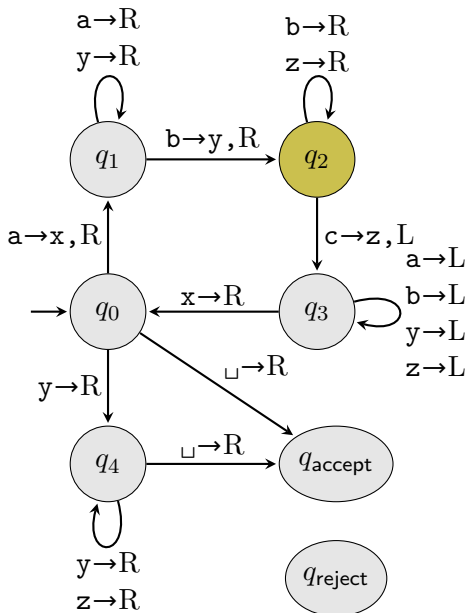
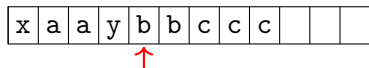




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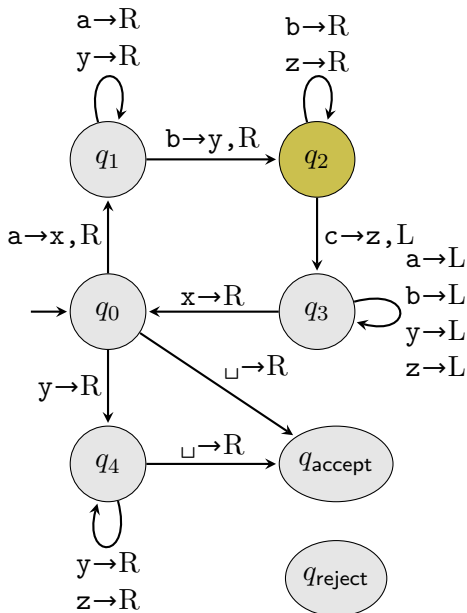
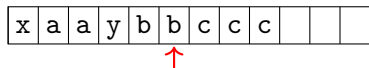
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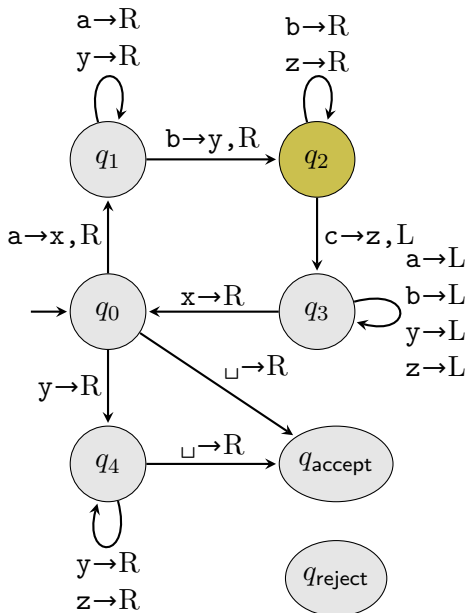
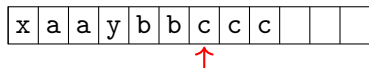
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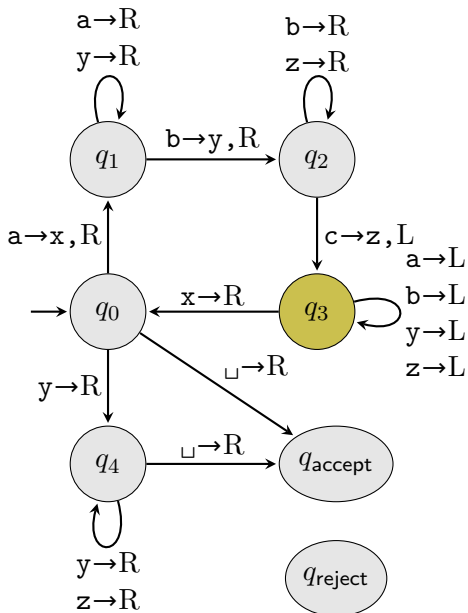
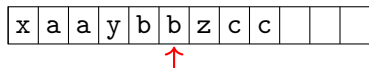
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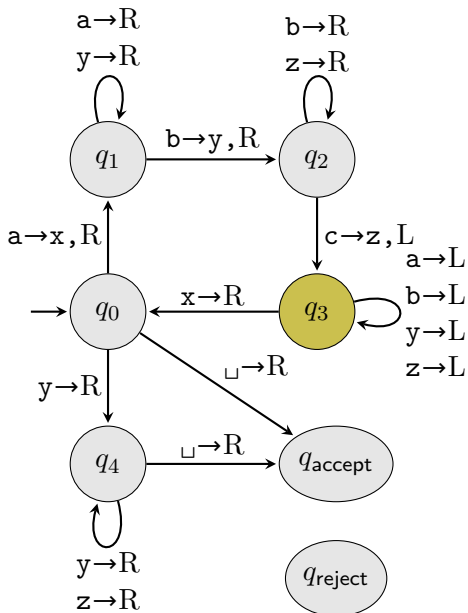
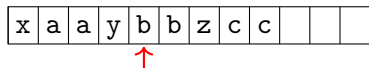
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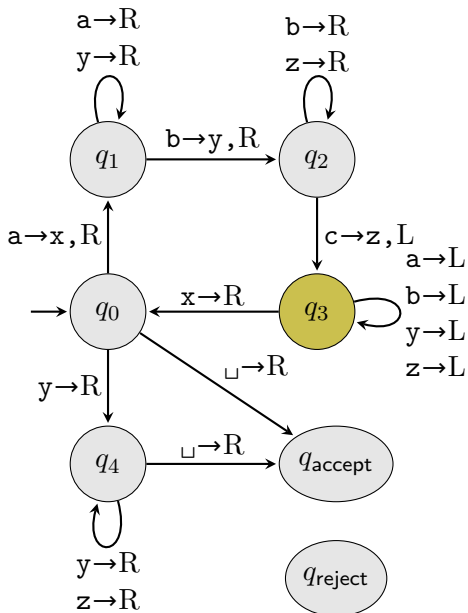
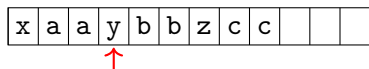
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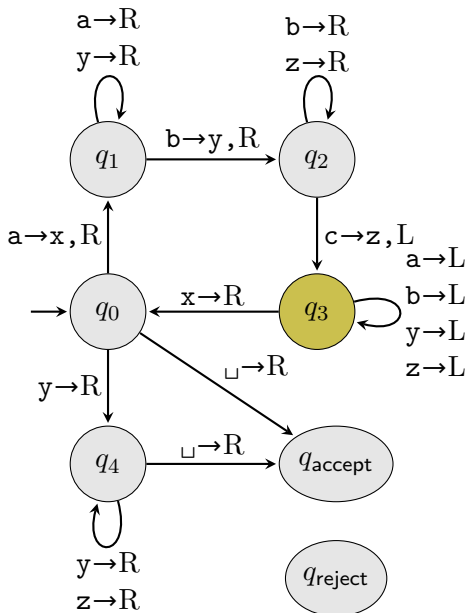
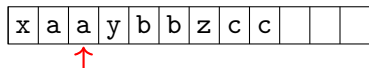
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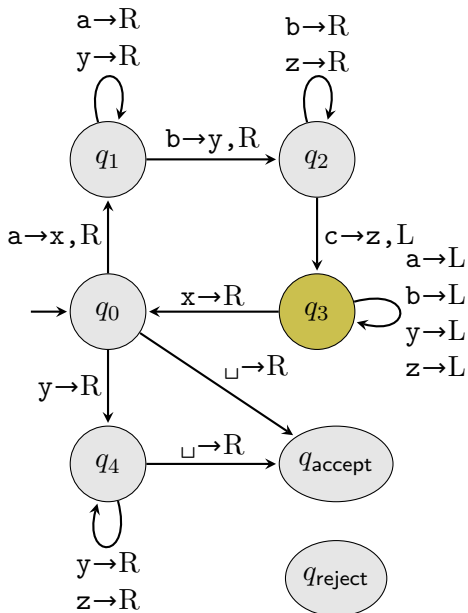
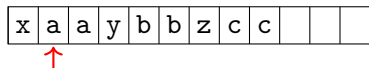
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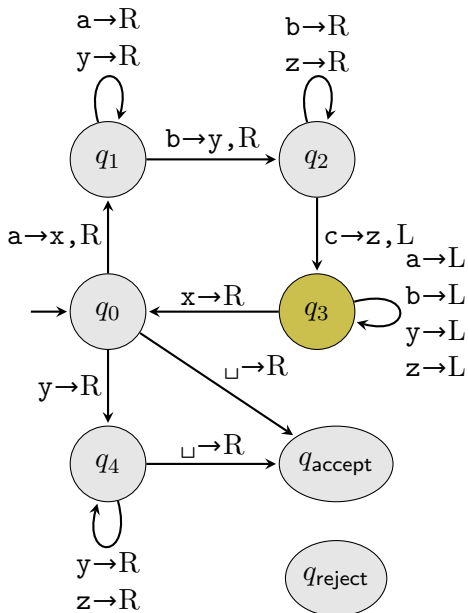
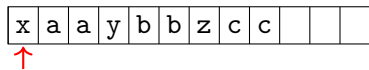




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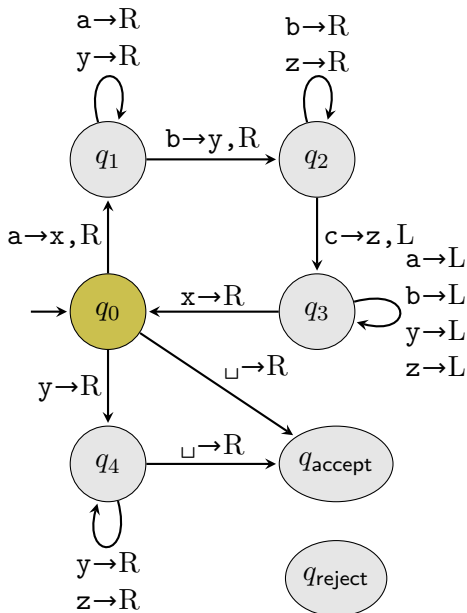
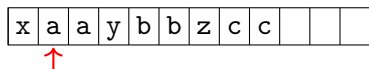
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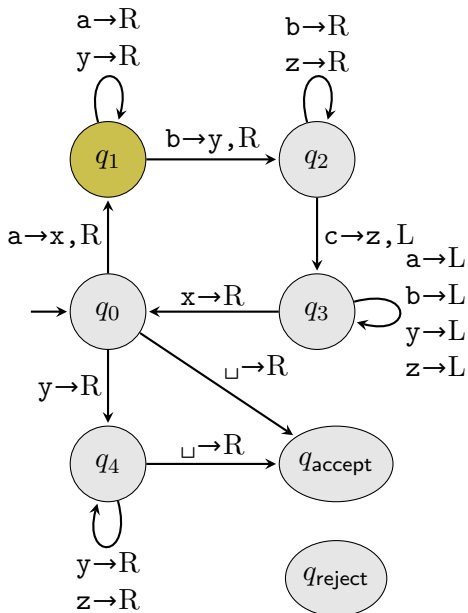
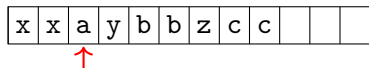
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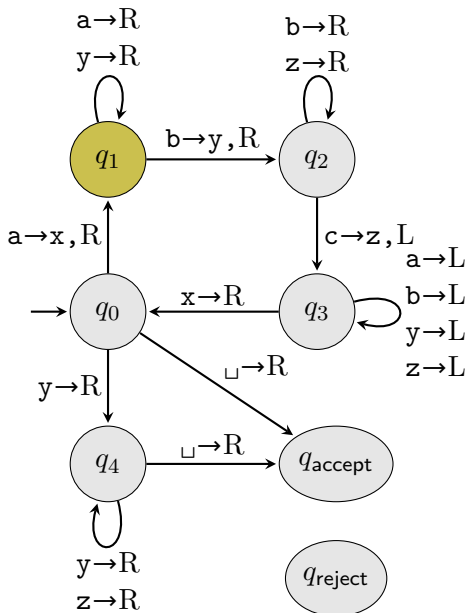
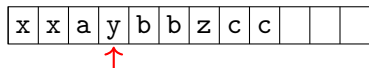
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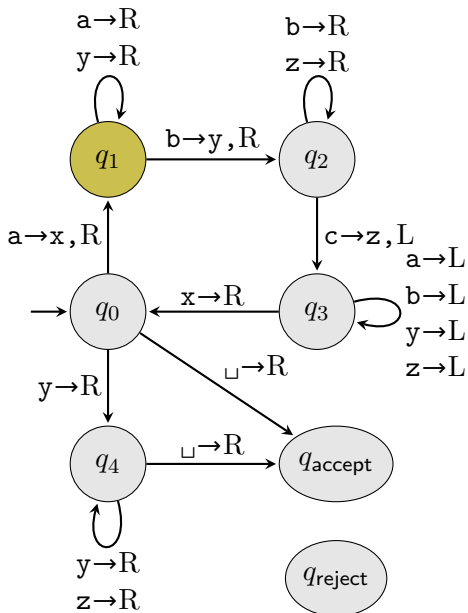
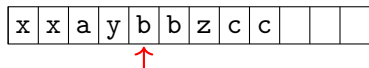
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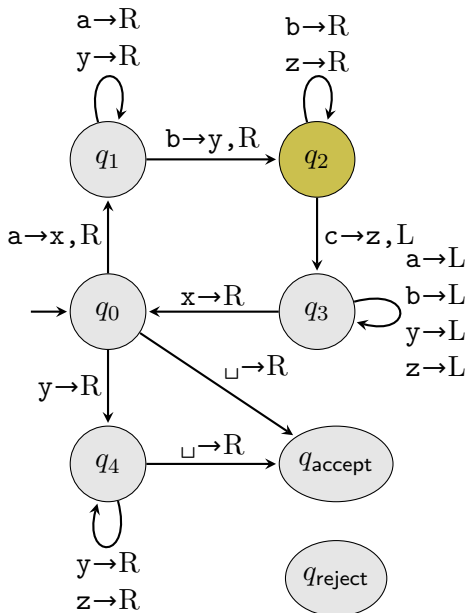
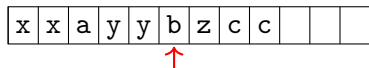
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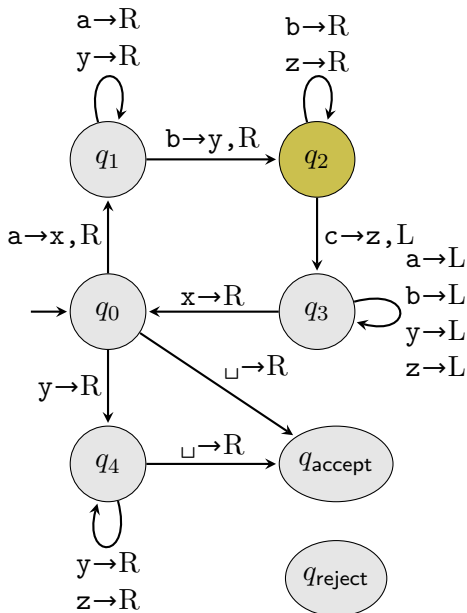
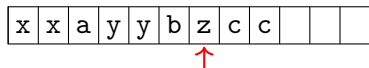
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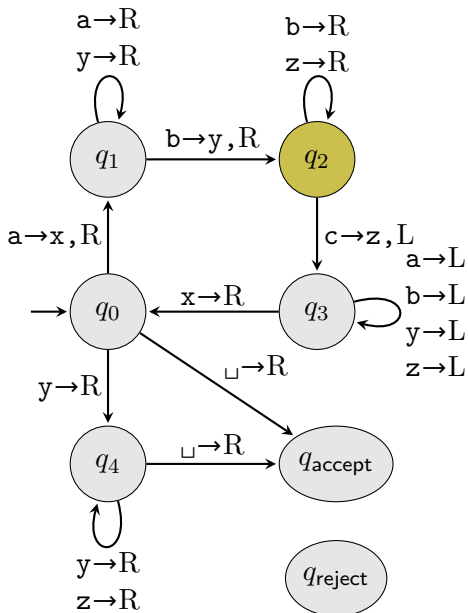
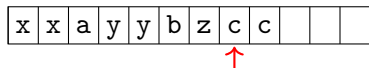
$w = aaabbbccc$



# Example

$$A = \{a^n b^n c^n \mid n \geq 0\}$$

$w = aaabbbccc$

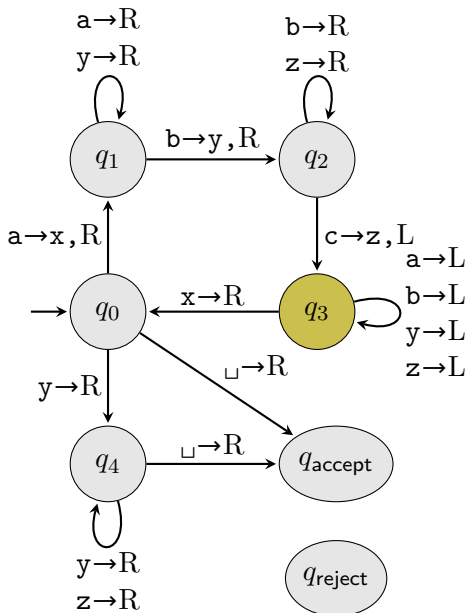
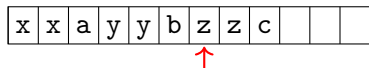




# Example

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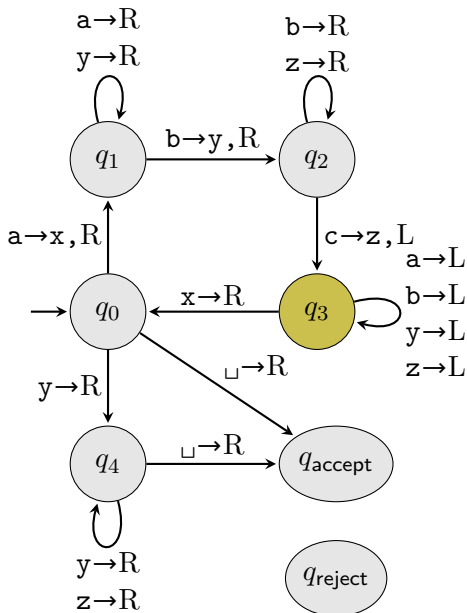
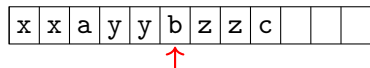
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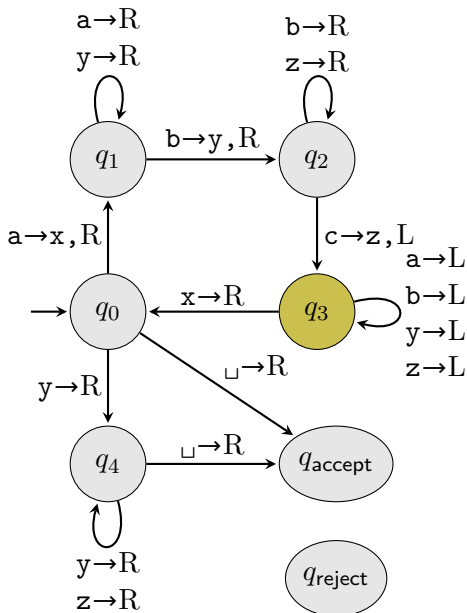
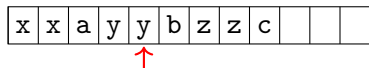
$w = aaabbbccc$



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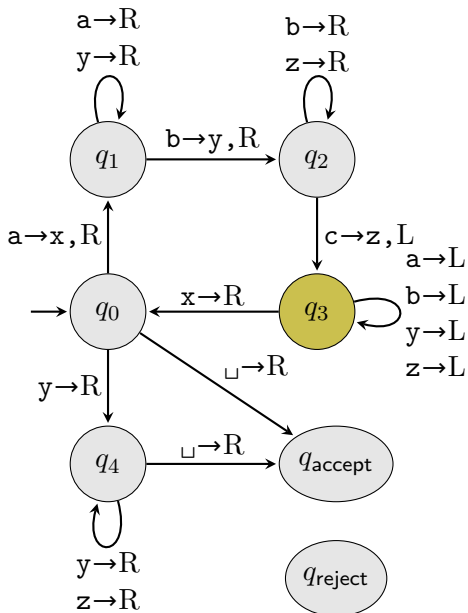
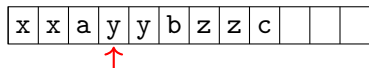
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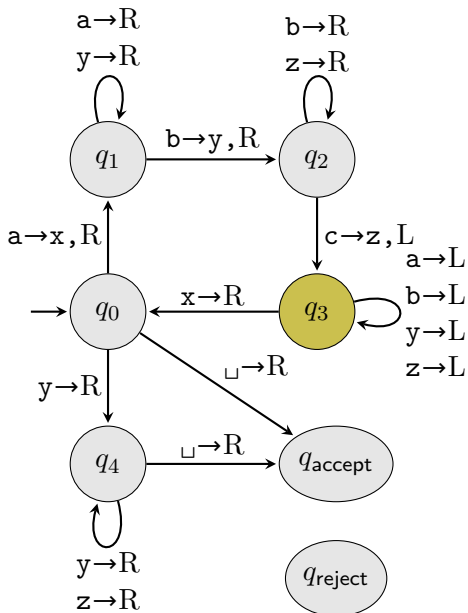
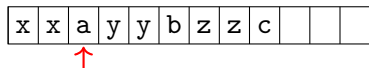
$w = aaabbbccc$



# Example

$$A = \{a^n b^n c^n \mid n \geq 0\}$$

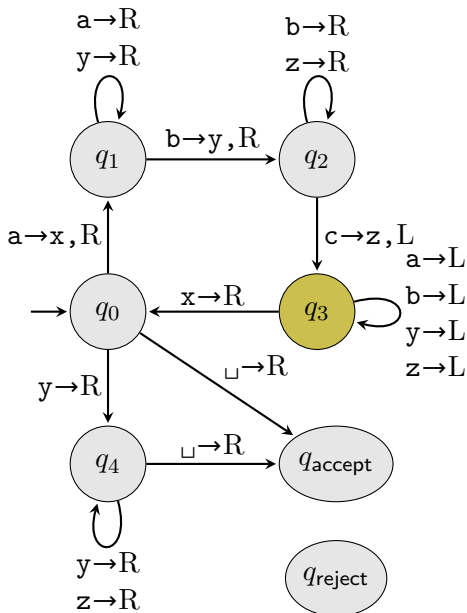
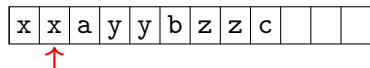
$w = aaabbbccc$



# Example

$$A = \{a^n b^n c^n \mid n \geq 0\}$$

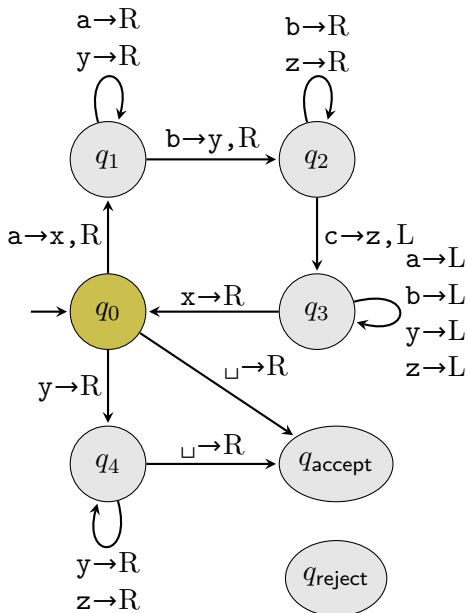
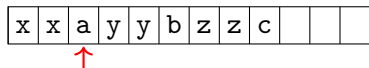
$w = aaabbbccc$



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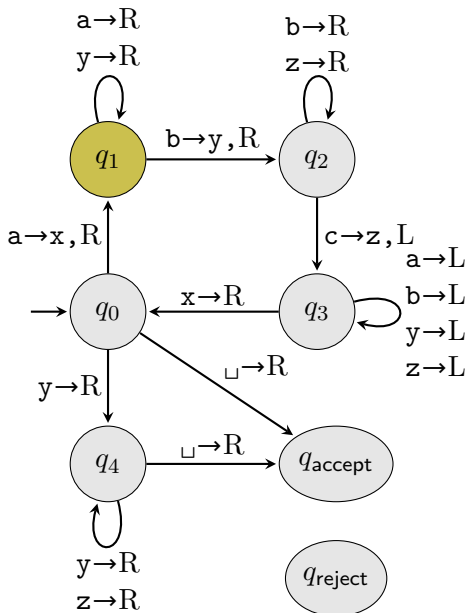
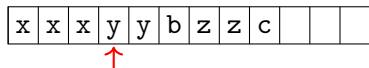
$w = aaabbbccc$



# Example

$$A = \{a^n b^n c^n \mid n \geq 0\}$$

$w = aaabbbccc$

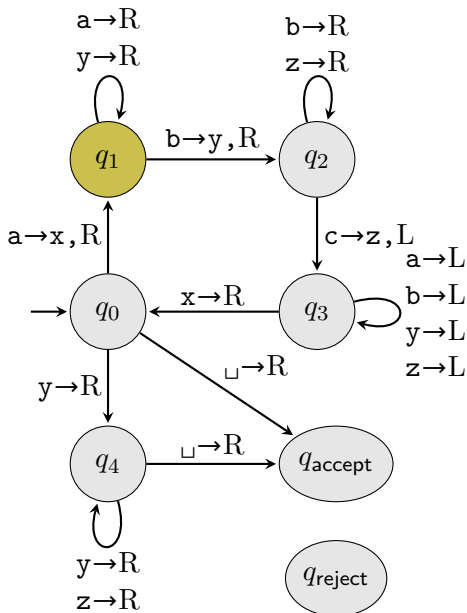
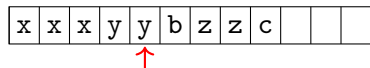




# Example

$$A = \{a^n b^n c^n \mid n \geq 0\}$$

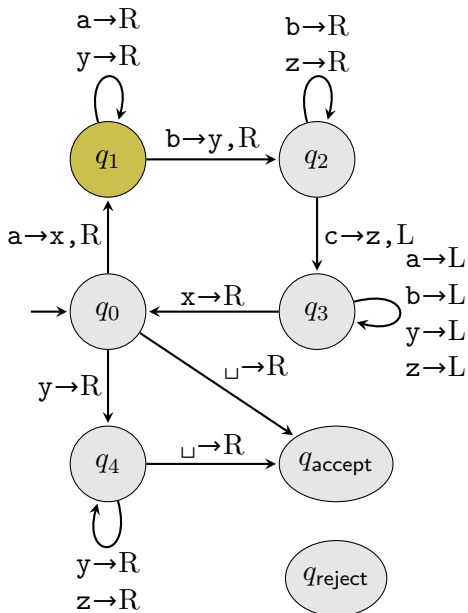
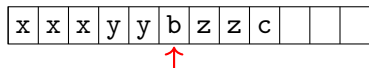
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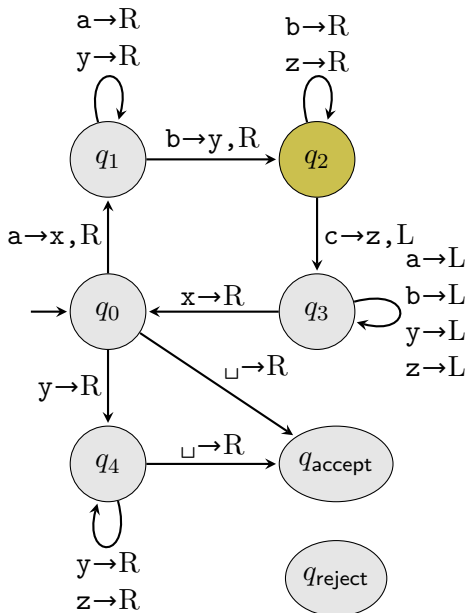
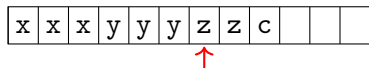
$w = aaabbbccc$



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$$A = \{a^n b^n c^n \mid n \geq 0\}$$

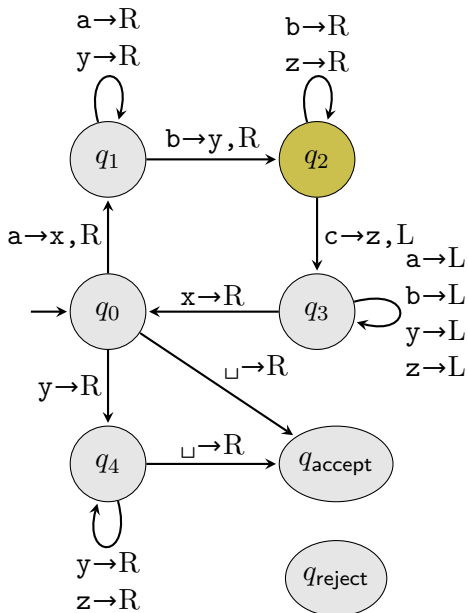
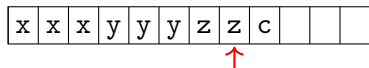
$w = aaabbbccc$



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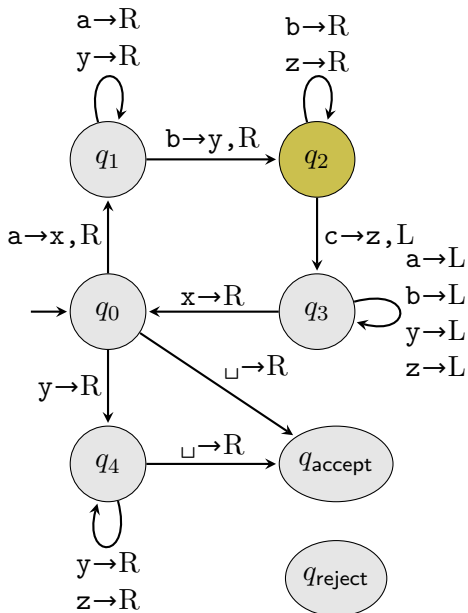
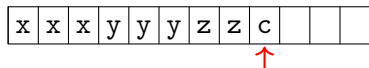
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# Example

$$A = \{a^n b^n c^n \mid n \geq 0\}$$

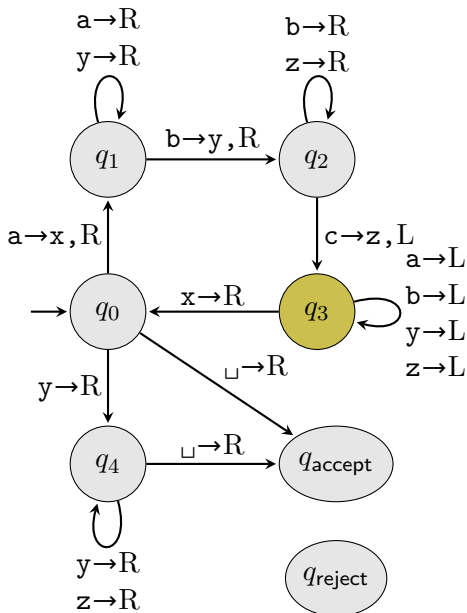
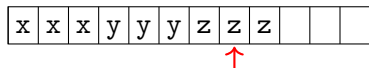
$w = aaabbbccc$



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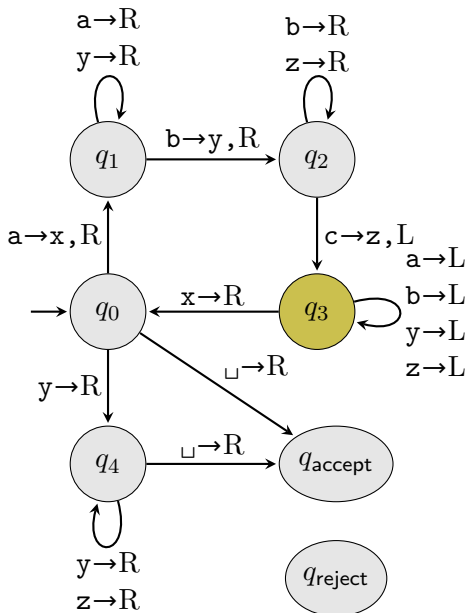
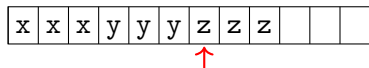
$w = aaabbbccc$



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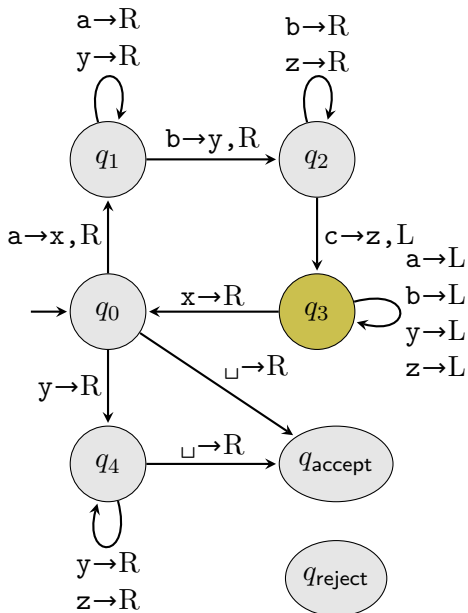
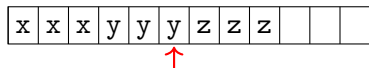
$w = aaabbbccc$



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$w = aaabbbccc$

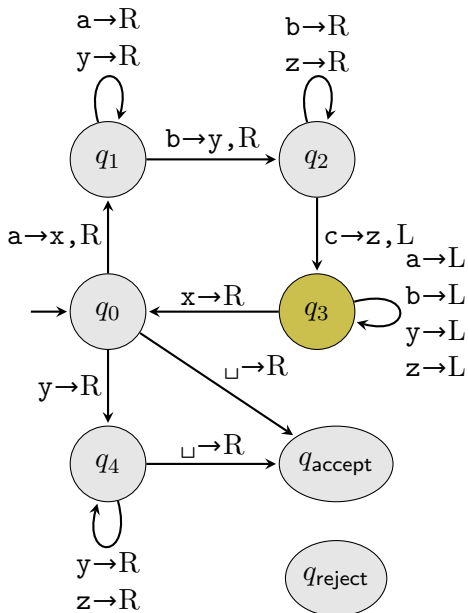
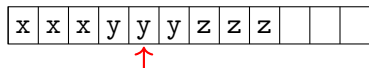




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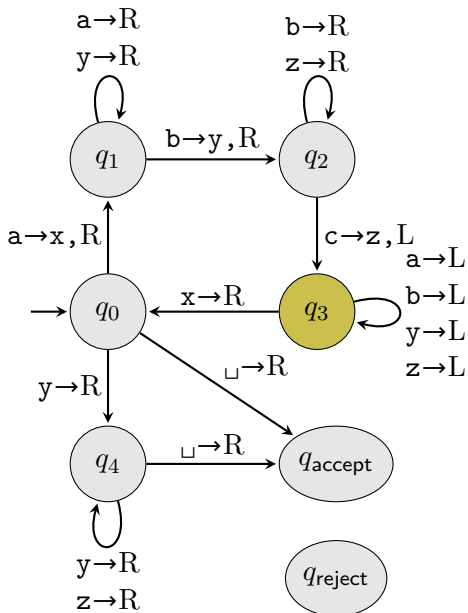
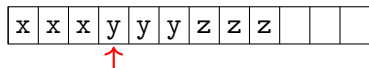
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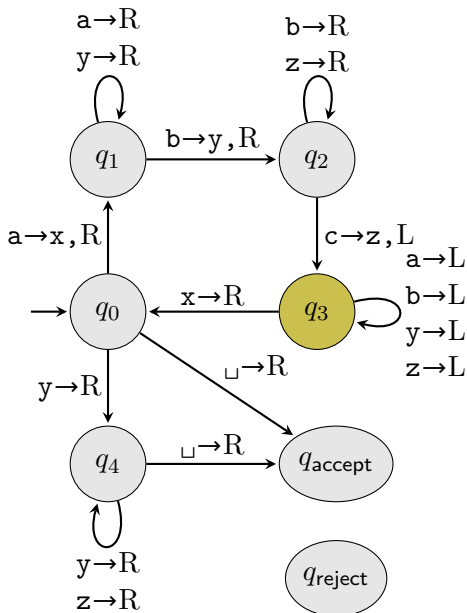
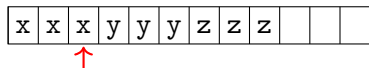
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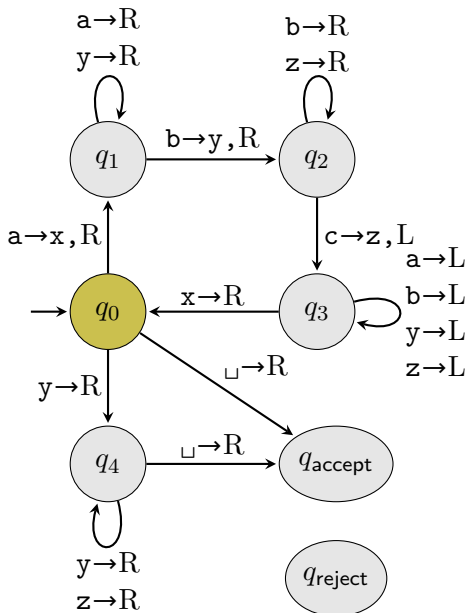
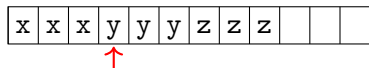
$w = aaabbbccc$



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$$A = \{a^n b^n c^n \mid n \geq 0\}$$

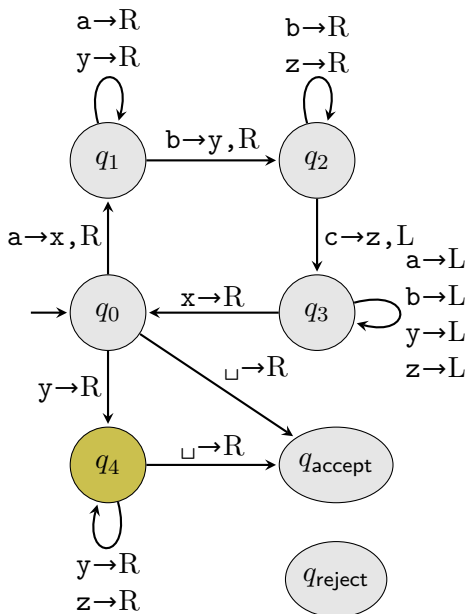
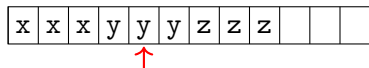
$w = aaabbbccc$



# Example

$$A = \{a^n b^n c^n \mid n \geq 0\}$$

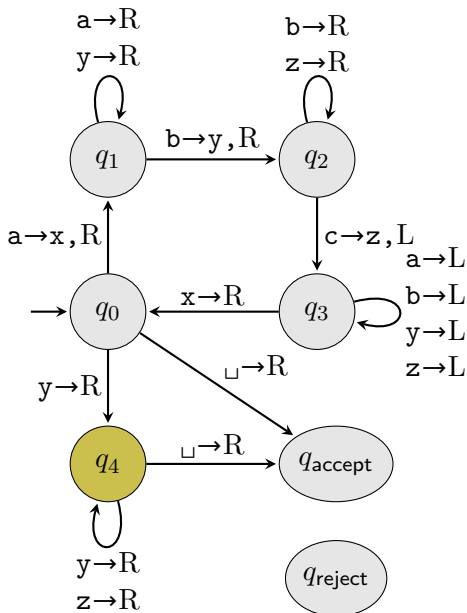
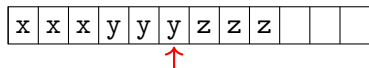
$w = aaabbbccc$



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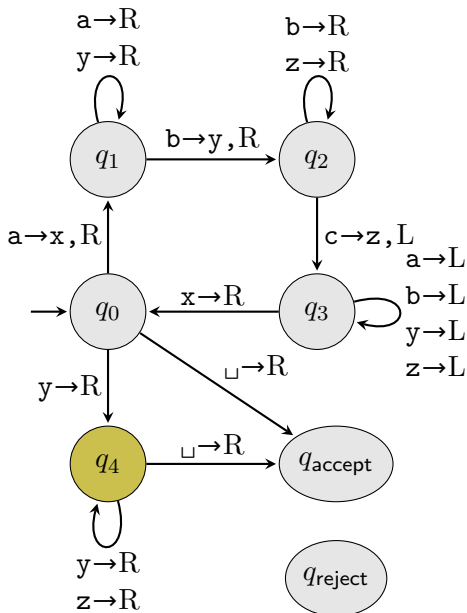
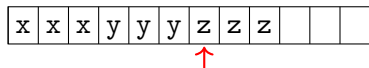
$w = aaabbbccc$



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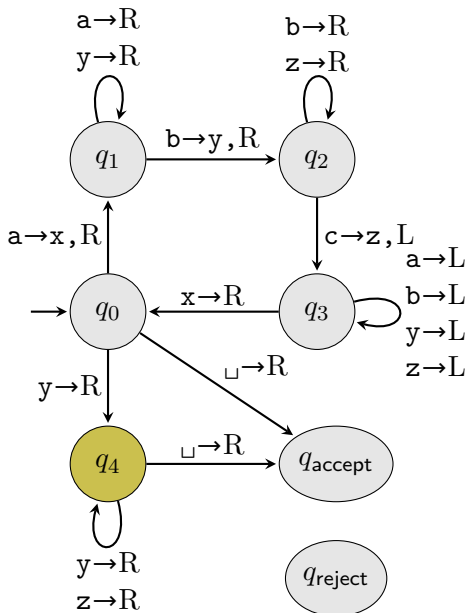
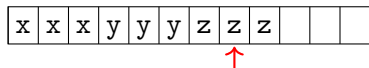
$w = aaabbbccc$



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$$A = \{a^n b^n c^n \mid n \geq 0\}$$

$w = aaabbbccc$

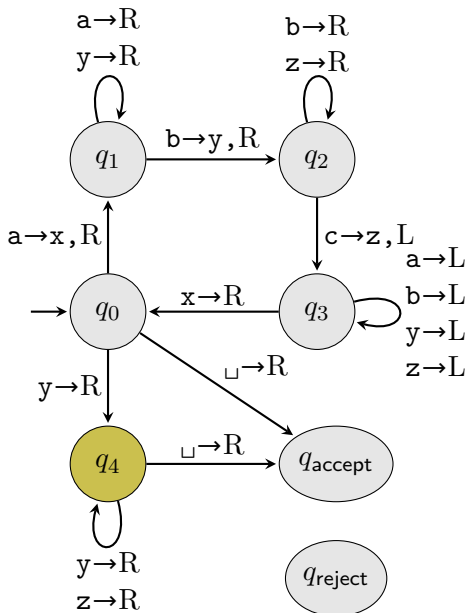
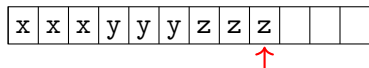




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$$A = \{a^n b^n c^n \mid n \geq 0\}$$

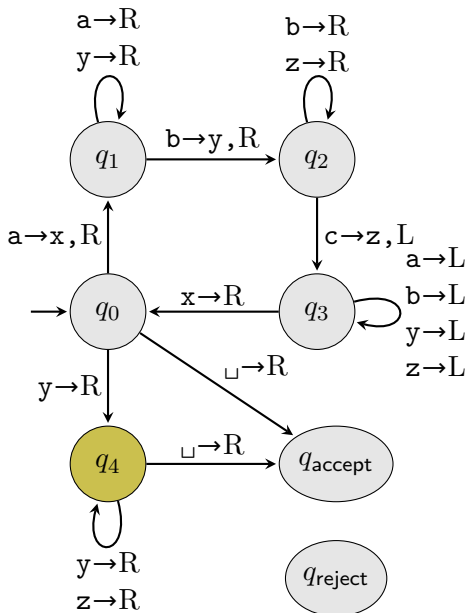
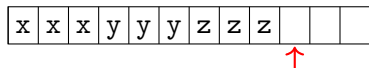
$w = aaabbbccc$



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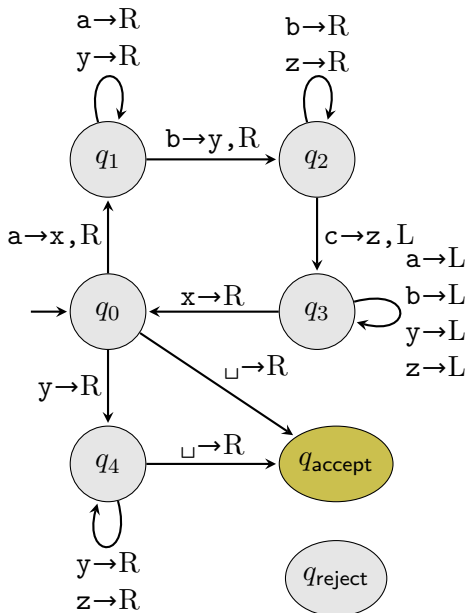
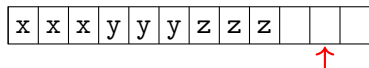
$w = aaabbbccc$



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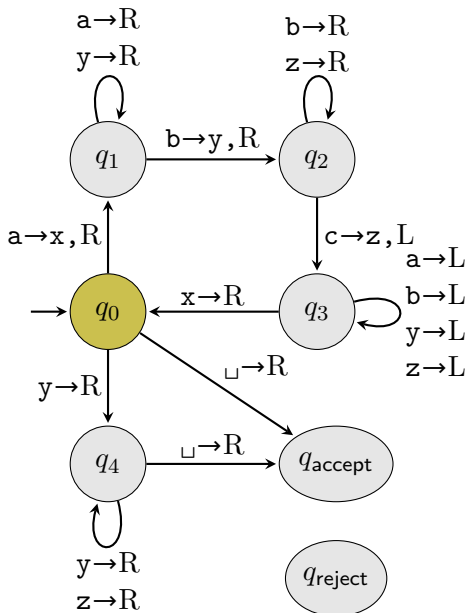
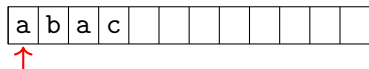
$w = aaabbbccc$



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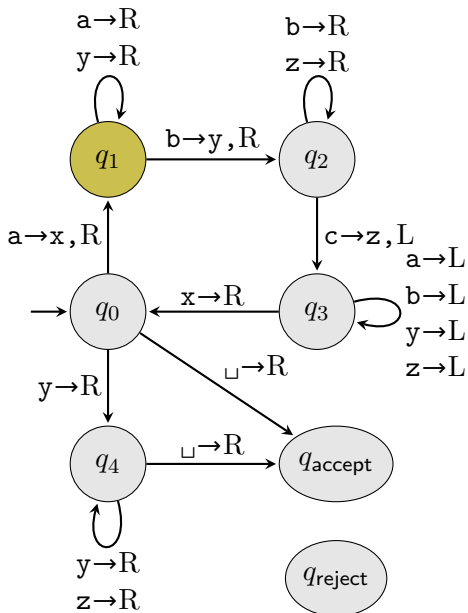
$w = abac$



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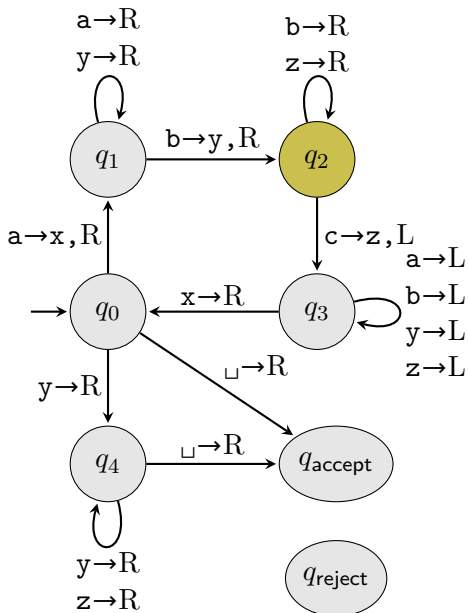
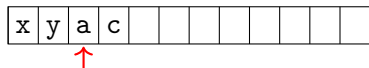
$w = abac$



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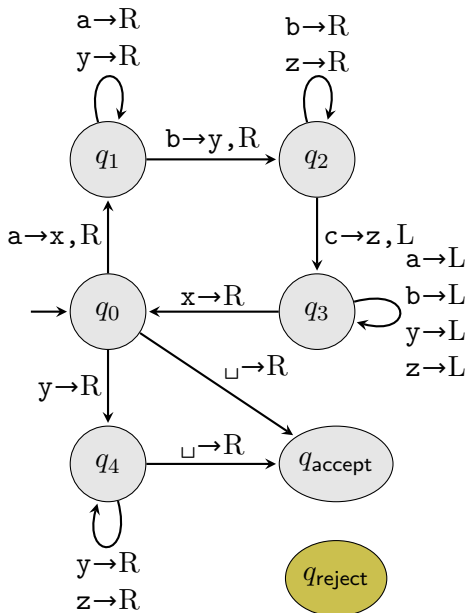
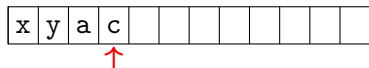
$w = abac$



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$$A = \{a^n b^n c^n \mid n \geq 0\}$$

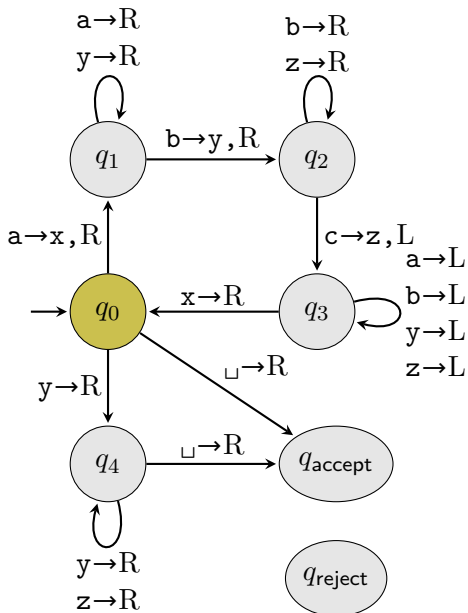
$w = abac$



# Example

$$A = \{a^n b^n c^n \mid n \geq 0\}$$

$$w = \varepsilon$$

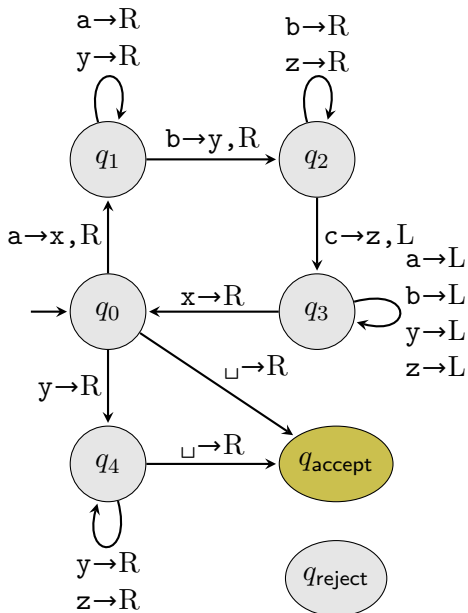




# Example

$$A = \{a^n b^n c^n \mid n \geq 0\}$$

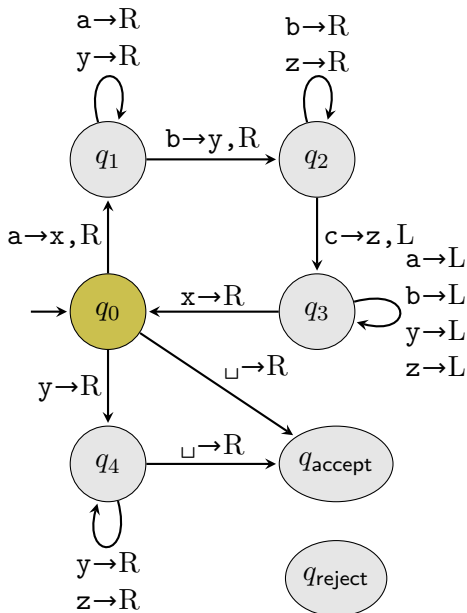
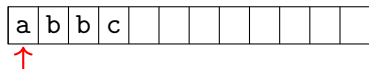
$$w = \varepsilon$$



# Example

$$A = \{a^n b^n c^n \mid n \geq 0\}$$

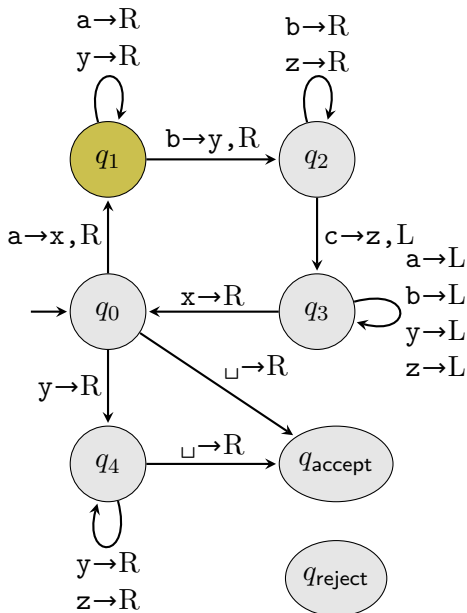
$w = abbc$



# Example

$$A = \{a^n b^n c^n \mid n \geq 0\}$$

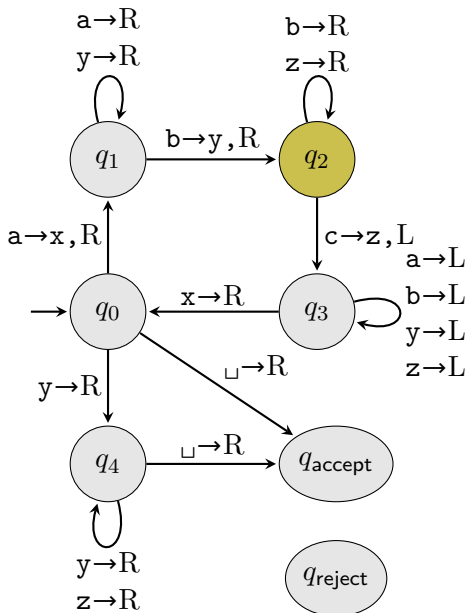
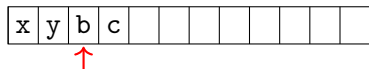
$w = abbc$



# Example

$$A = \{a^n b^n c^n \mid n \geq 0\}$$

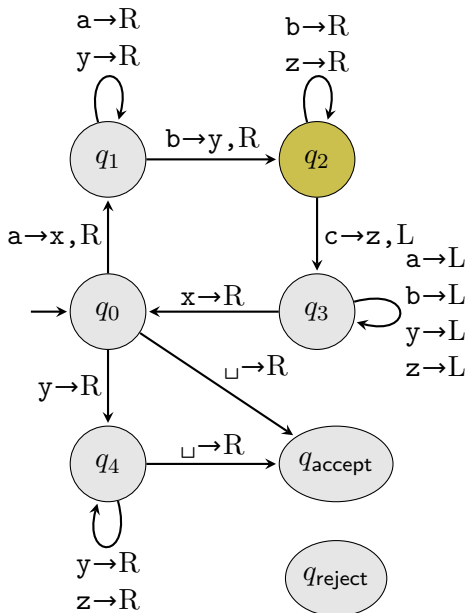
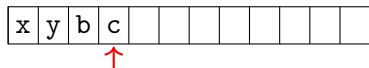
$w = abbc$



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$$A = \{a^n b^n c^n \mid n \geq 0\}$$

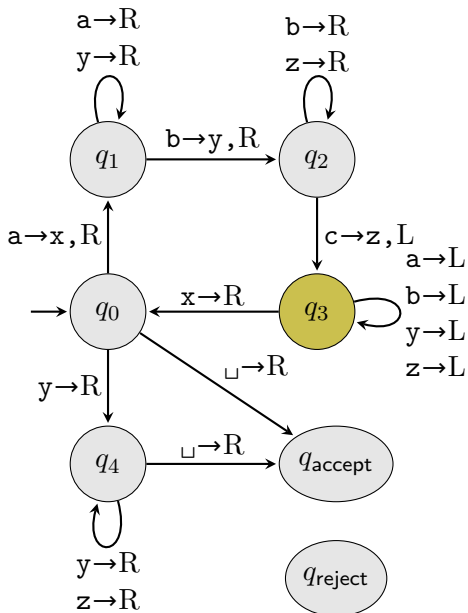
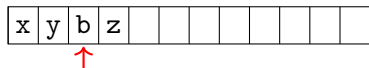
$w = abbc$



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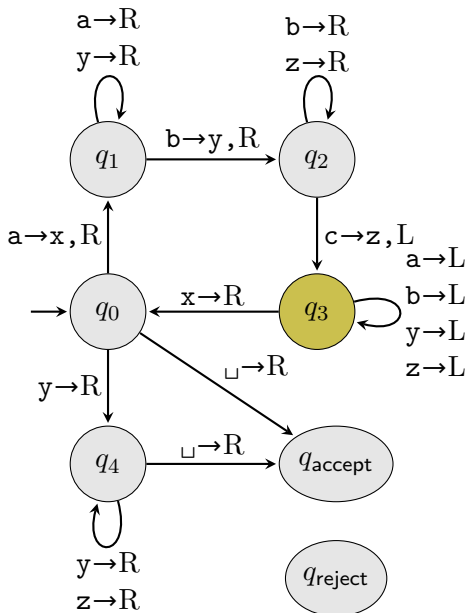
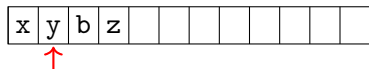
$w = abbc$



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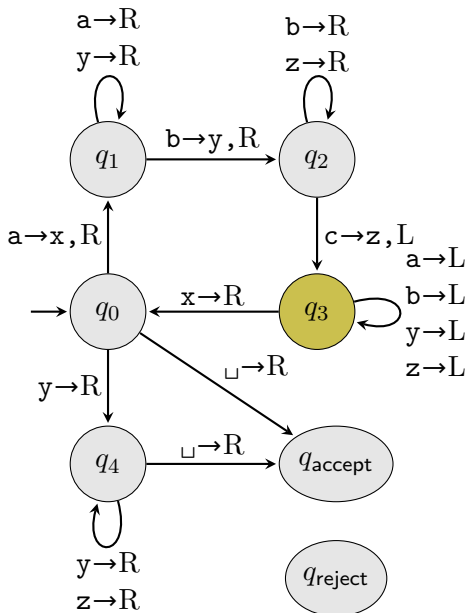
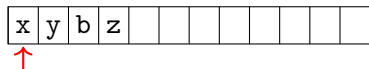
$w = abbc$



# Example

$$A = \{a^n b^n c^n \mid n \geq 0\}$$

$w = abbc$

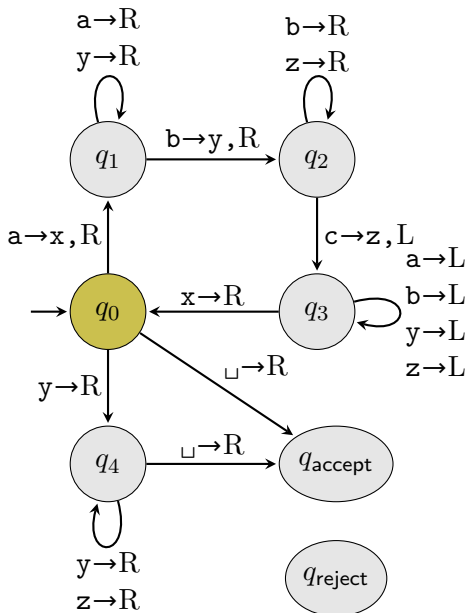
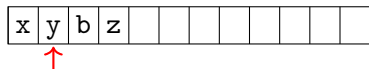




# Example

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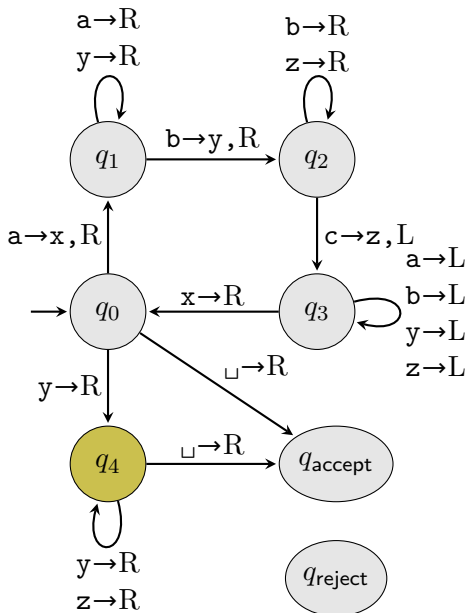
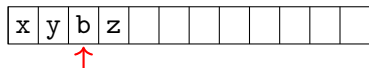
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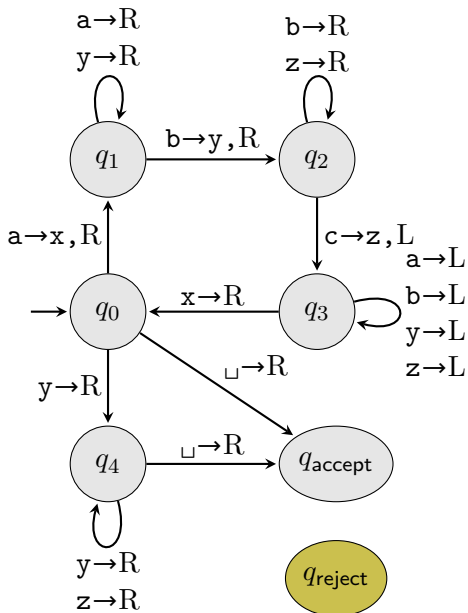
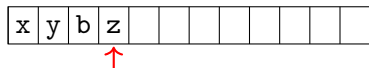
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# Mathematically

A TM is a 7-tuple  $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$  where

$Q$  finite **set of states**

$\Sigma$  finite **input alphabet**  $\sqcup \notin \Sigma$

$\Gamma$  finite **tape alphabet**  $\Sigma \subseteq \Gamma$  and  $\sqcup \in \Gamma$

$\delta$  **transition function**

$q_0$  **initial state**  $q_0 \in Q$

$q_{\text{accept}}$  **single accepting state**  $q_{\text{accept}} \in Q$

$q_{\text{reject}}$  **single rejecting state**  $q_{\text{reject}} \in Q$

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For every other state  $q$  and tape symbol  $s \in \Gamma$  (including  $s = \sqcup$ ),

$$(r, t, D) = \delta(q, s)$$

means: if the cell under the tape head is  $s$ , the TM

- ① changes to state  $r$ ;
- ② replaces  $s$  on the tape with  $t$ ; and
- ③ moves the tape head in the direction  $D \in \{L, R\}$

## Turing-recognizable languages

The **language** of a TM  $M$ , denoted  $L(M)$  is the set of strings accepted by  $M$   
Similarly,  $M$  **recognizes** a language  $A$  if  $L(M) = A$

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A language  $A$  is **Turing-recognizable** (abbreviated RE<sup>1</sup>) if some TM recognizes it

**Example:**

$\{a^n b^n c^n \mid n \geq 0\}$  is RE because we constructed a TM to recognize it

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## Decidable languages

For a given string  $w \in \Sigma^*$ , if TM  $M$  either accepts or rejects the string, we say that  $M$  **halts** on  $w$

Otherwise, we say  $M$  **loops** on  $w$  because it must have entered an infinite loop

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A TM  $M$  is called a **decider** if  $M$  halts on every input

A decider that recognizes a language is said to **decide** it.

I.e., “TM  $M$  decides  $A$ ” means “ $L(M) = A$  and on every input  $M$  halts”

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A language is **decidable**<sup>2</sup> if some TM decides it

**Example:**

$\{a^n b^n c^n \mid n \geq 0\}$  is decidable because we built a decider that recognizes it

Equivalently, it's decidable because we built a TM that decides it

---

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## Decidability is a property of the language

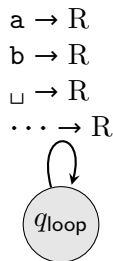
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Given any TM  $M$ , we can add a new state to  $M$  that contains a loop that moves the head right for every  $t \in \Gamma$  and replace all transitions to  $q_{\text{reject}}$  to this looping state

Now, instead of rejecting strings,  $M$  will loop.

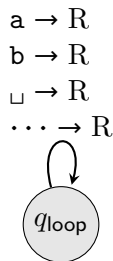


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If we perform this modification to our decider  $M$  for  $A = \{a^n b^n c^n \mid n \geq 0\}$ , we'll get a TM  $M'$  such that  $L(M') = A$  that doesn't halt on, e.g.,  $w = ba$  and thus isn't a decider

Nevertheless,  $A$  is decidable

## Regular languages are decidable

Given a DFA  $M = (Q, \Sigma, \delta, q_0, F)$ , we can construct a TM  $M' = (Q', \Sigma, \Gamma, \delta', q_0, q_{\text{accept}}, q_{\text{reject}})$  where

$$Q' = Q \cup \{q_{\text{accept}}, q_{\text{reject}}\}$$

$$\Gamma = \Sigma \cup \{\sqcup\}$$

$$\delta'(q, t) = \begin{cases} (\delta(q, t), t, \text{R}) & \text{if } t \in \Sigma \\ (q_{\text{accept}}, \sqcup, \text{R}) & \text{if } t = \sqcup \text{ and } q \in F \\ (q_{\text{reject}}, \sqcup, \text{R}) & \text{if } t = \sqcup \text{ and } q \notin F \end{cases}$$



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When  $M'$  runs on input  $w$ , it moves through the input one symbol at a time because it always moves to the right.

Once  $M'$  reaches the end of the input, the cell under the tape head will contain  $\sqcup$  and if  $M'$  is in a state in  $F$ , it will transition to  $q_{\text{accept}}$ , otherwise it will transition to  $q_{\text{reject}}$ .

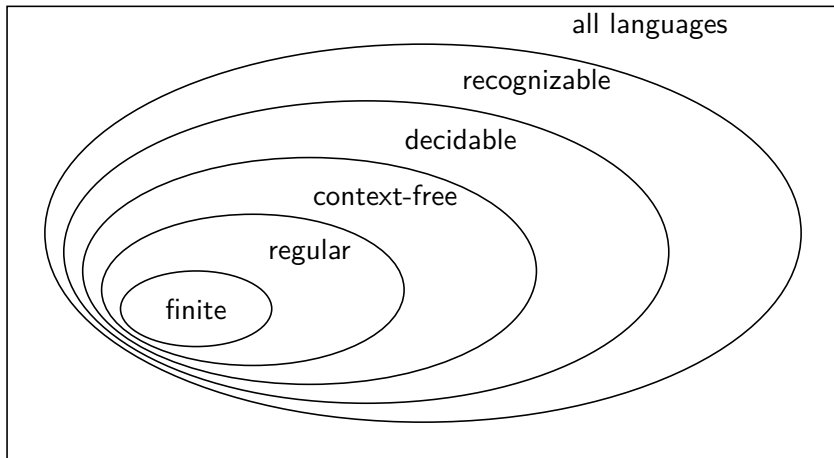
Thus  $L(M') = L(M)$ . Since  $M'$  always moves right and halts after the first  $\sqcup$ , it halts on every input and thus is a decider



# Relationship between languages

Later, we'll prove

- every CFL is decidable
- there exist languages that aren't decidable but are Turing-recognizable
- there exist languages that aren't Turing-recognizable



# TM configurations

A **configuration** of a TM is a triple  $uqv$  where

$u \in \Gamma^*$  is the tape to the left of the head

$q \in Q$  is the current state

$v \in \Gamma^*$  is the portion of tape under and to the right of the head up to at least the last nonblank cell

The tape itself is  $uv$  plus an infinite number of blanks to the right (note that as this is infinite, it's not a string)

## Computation of a TM

TM  $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$  accepts  $w \in \Sigma^*$ , if there is a sequence of configurations  $c_0, c_1, \dots, c_n$  where  $c_i = u_i n_i v_i$  such that

- 1  $c_0 = u_0 r_0 v_0$  is the start configuration:  $u_0 = \varepsilon$ ,  $r_0 = q_0$ , and  $v_0 = w$ ;
- 2 for  $i > 0$ ,  $c_i$  follows from  $c_{i-1}$  according to  $\delta$ ; and
- 3  $c_n$  is an accepting configuration, that is,  $r_n = q_{\text{accept}}$

Moving from  $c_{i-1}$  to  $c_i$  is slightly tricky to write down but conceptually simple

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- If  $D = L$  and  $u_{i-1} = \varepsilon$ , then  $u_i = \varepsilon$ ,  $r_i = r$ , and  $v_i = tv$

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- If  $D = L$  and  $u_{i-1} = \varepsilon$ , then  $u_i = \varepsilon$ ,  $r_i = r$ , and  $v_i = tv$
- If  $D = L$  and  $u_{i-1} = ux$  for  $x \in \Gamma$  and  $u \in \Gamma^*$ , then  $u_i = u$ ,  $r_i = r$ , and  $v_i = xtv$



# Example configurations

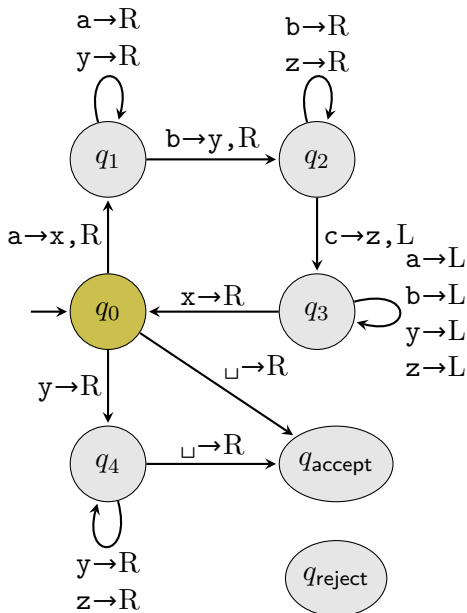
$$A = \{a^n b^n c^n \mid n \geq 0\}$$

$$w = abc$$



Configurations:

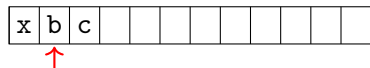
**0**  $q_0 abc$



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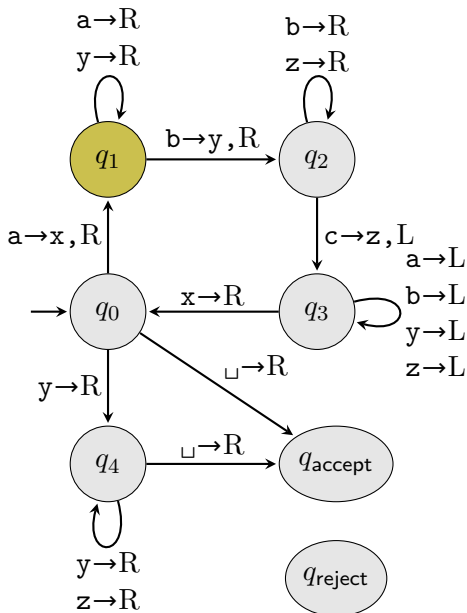
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Configurations:

0  $q_0 abc$

1  $xq_1 bc$



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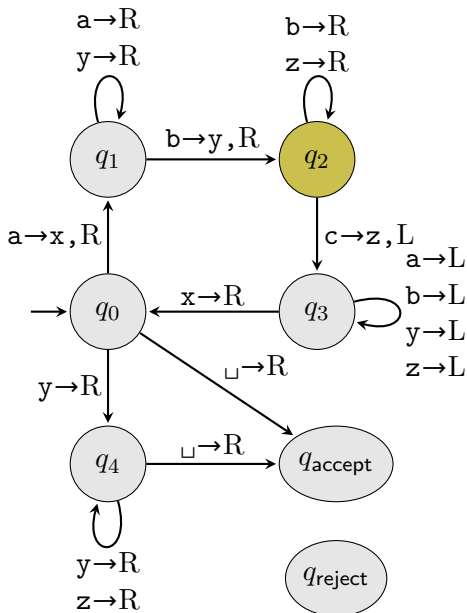
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Configurations:

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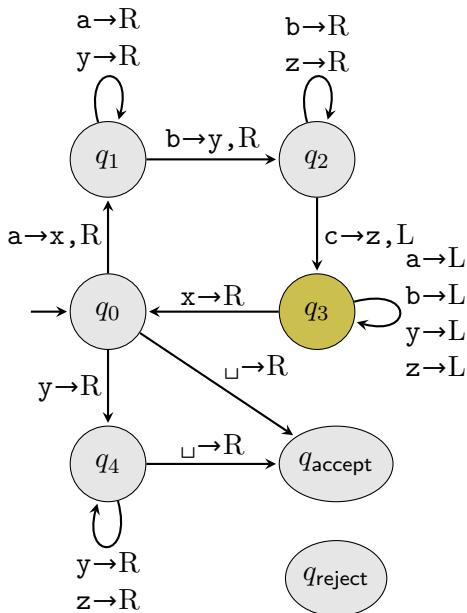
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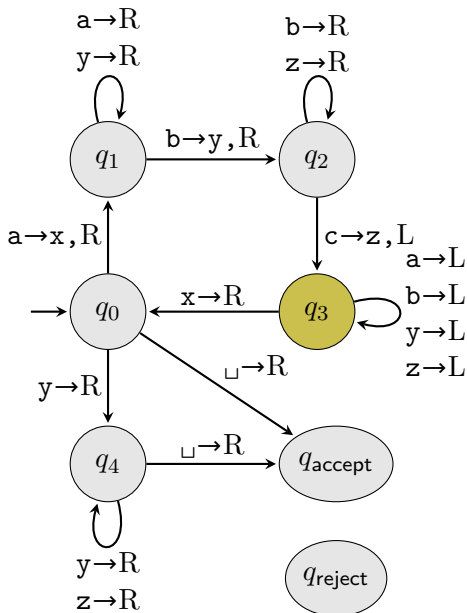
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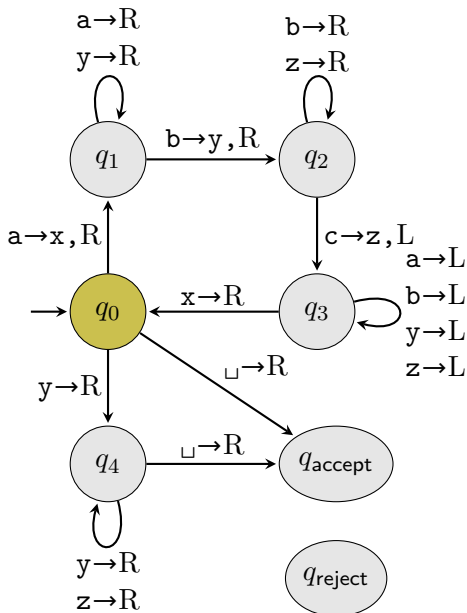
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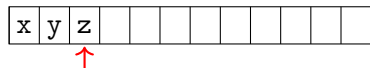
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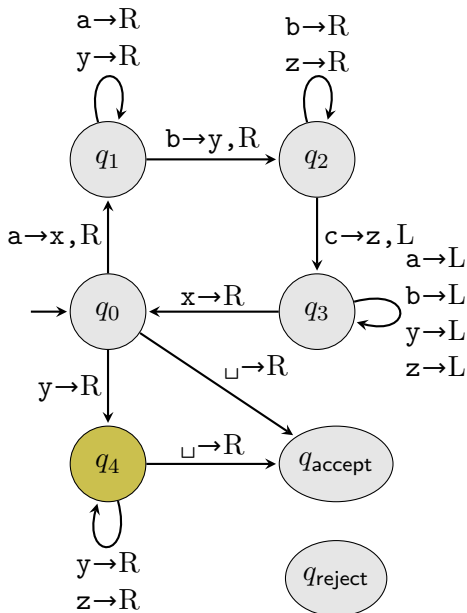
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Configurations:

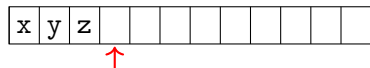
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- ⑤  $q_3 xyz$
- ⑥  $xq_0 yz$
- ⑦  $xyq_4 z$



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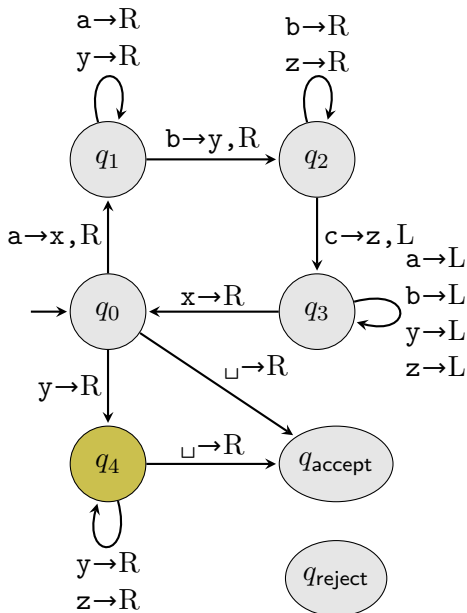
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Configurations:

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- ⑥  $xq_0 yz$
- ⑦  $xyzq_4 \sqcup$

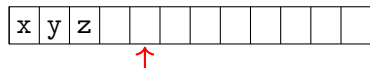




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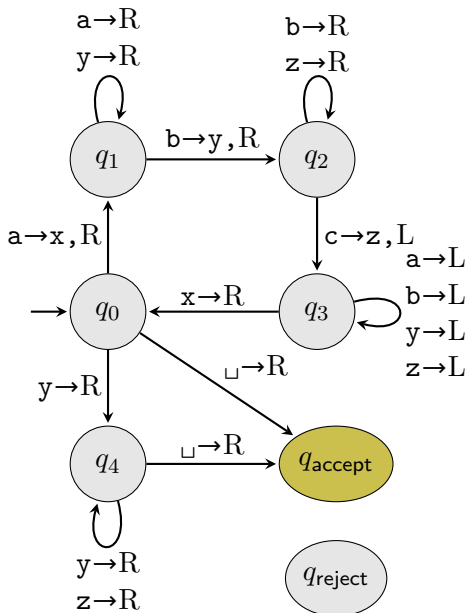
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$$w = abc$$



Configurations:

- 0  $q_0 abc$
- 1  $xq_1 bc$
- 2  $xyq_2 c$
- 3  $xq_3 yz$
- 4  $q_3 xyz$
- 5  $xq_0 yz$
- 6  $xyq_4 z$
- 7  $xyzq_4 \sqcup$
- 8  $xyz \sqcup q_{\text{accept}} \sqcup$



# Multiplication

$$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$$

High-level description of a TM to decide  $B$ :

- Cross an  $a$  off (i.e., replace it with something) and then for each  $b$ , cross off a  $c$  by moving back and forth between the  $b$ s and  $c$ s, crossing each off in turn (by replacing  $b$  with  $y$  and  $c$  with  $z$ )

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- Once all the  $b$ s (and their corresponding  $c$ s) have been crossed off, restore all of the  $b$
- Repeat steps 1 and 2 until all of the  $a$ s have been crossed off

## Multiplication

$$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$$

High-level description of a TM to decide  $B$ :

- Cross an  $a$  off (i.e., replace it with something) and then for each  $b$ , cross off a  $c$  by moving back and forth between the  $b$ s and  $c$ s, crossing each off in turn (by replacing  $b$  with  $y$  and  $c$  with  $z$ )
- Once all the  $b$ s (and their corresponding  $c$ s) have been crossed off, restore all of the  $b$
- Repeat steps 1 and 2 until all of the  $a$ s have been crossed off
- If any  $c$ s remain or if at any point the input was not in the correct order, reject. Otherwise accept.

## Multiplication implementation-level description

$$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$$

Implementation-level description of a TM to decide  $B$ :

- 1 As always, the TM starts with the input on the tape with the head pointing to the first position

## Multiplication implementation-level description

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Implementation-level description of a TM to decide  $B$ :

- 1 As always, the TM starts with the input on the tape with the head pointing to the first position
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Implementation-level description of a TM to decide  $B$ :

- 1 As always, the TM starts with the input on the tape with the head pointing to the first position
- 2 If the cell under the head is not a, reject, otherwise replace a with x and scan right past all as
- 3 If the cell under the head is not b, reject, otherwise replace b with y and scan right past all bs and zs (which will be the crossed off cs)



## Multiplication implementation-level description

$$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$$

Implementation-level description of a TM to decide  $B$ :

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- 3 If the cell under the head is not b, reject, otherwise replace b with y and scan right past all bs and zs (which will be the crossed off cs)
- 4 If the cell under the head is not c, reject, otherwise replace c with z and scan left until there is a y (the most recently crossed off b)

## Multiplication implementation-level description

$$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$$

Implementation-level description of a TM to decide  $B$ :

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- 4 If the cell under the head is not c, reject, otherwise replace c with z and scan left until there is a y (the most recently crossed off b)
- 5 If the cell under the head is b, goto step 3

## Multiplication implementation-level description

$$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$$

Implementation-level description of a TM to decide  $B$ :

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- 5 If the cell under the head is b, goto step 3
- 6 Otherwise, the cell is a z so scan left, replacing each y with b

## Multiplication implementation-level description

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- 5 If the cell under the head is b, goto step 3
- 6 Otherwise, the cell is a z so scan left, replacing each y with b
- 7 If the cell under the head is a, scan left to the first a (it's just right of the last x) and return to step 1

## Multiplication implementation-level description

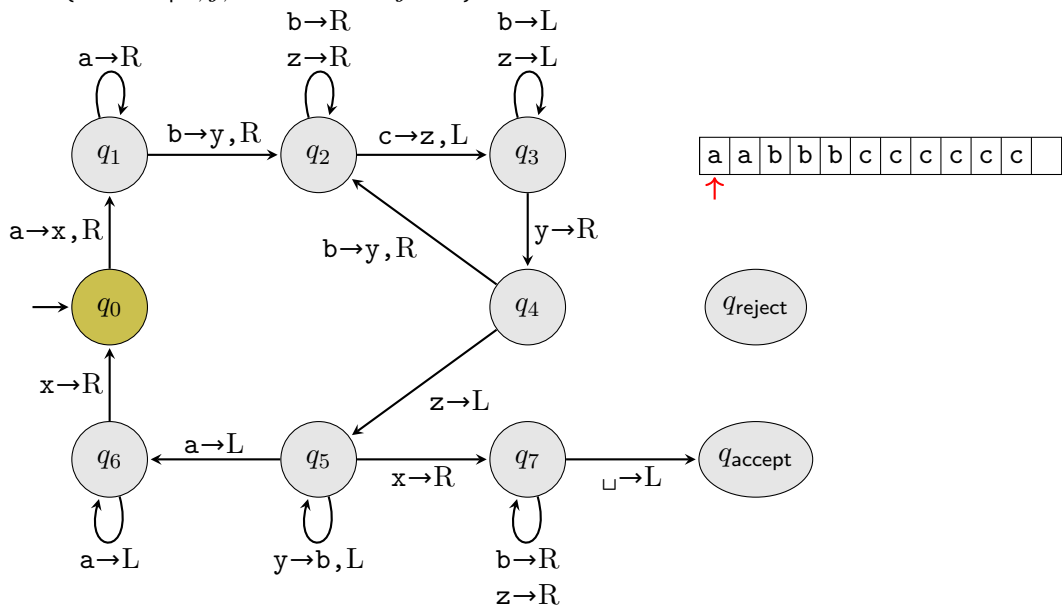
$$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$$

Implementation-level description of a TM to decide  $B$ :

- 1 As always, the TM starts with the input on the tape with the head pointing to the first position
- 2 If the cell under the head is not a, reject, otherwise replace a with x and scan right past all as
- 3 If the cell under the head is not b, reject, otherwise replace b with y and scan right past all bs and zs (which will be the crossed off cs)
- 4 If the cell under the head is not c, reject, otherwise replace c with z and scan left until there is a y (the most recently crossed off b)
- 5 If the cell under the head is b, goto step 3
- 6 Otherwise, the cell is a z so scan left, replacing each y with b
- 7 If the cell under the head is a, scan left to the first a (it's just right of the last x) and return to step 1
- 8 Otherwise, move right (past the x) and then scan left past all of the bs and zs. If the cell under the head is blank (because all cs have been replaced with zs), accept, otherwise reject

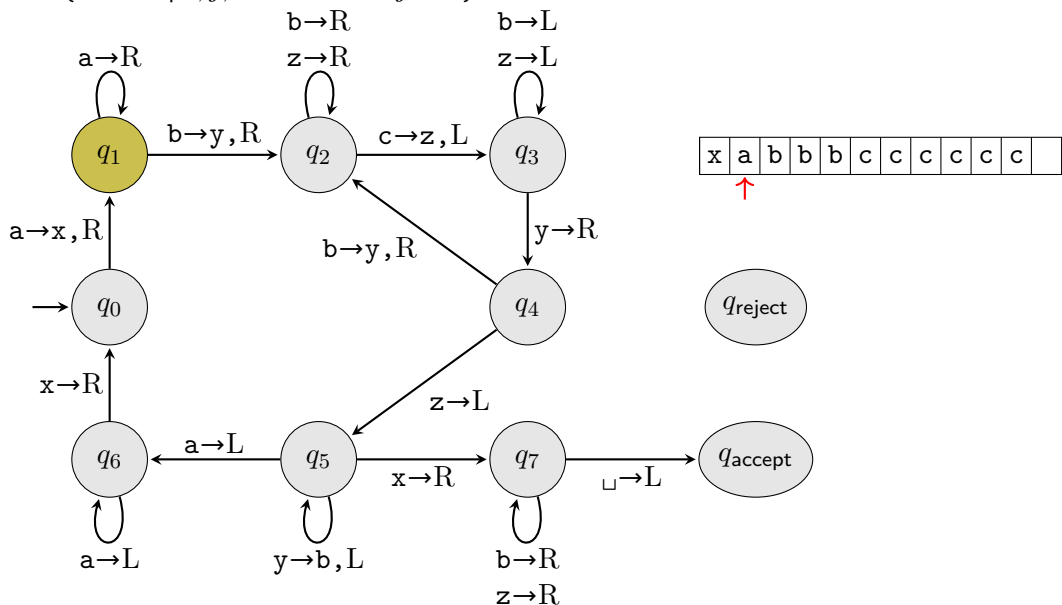
# Multiplication TM

$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = \text{aabbbccccc}$



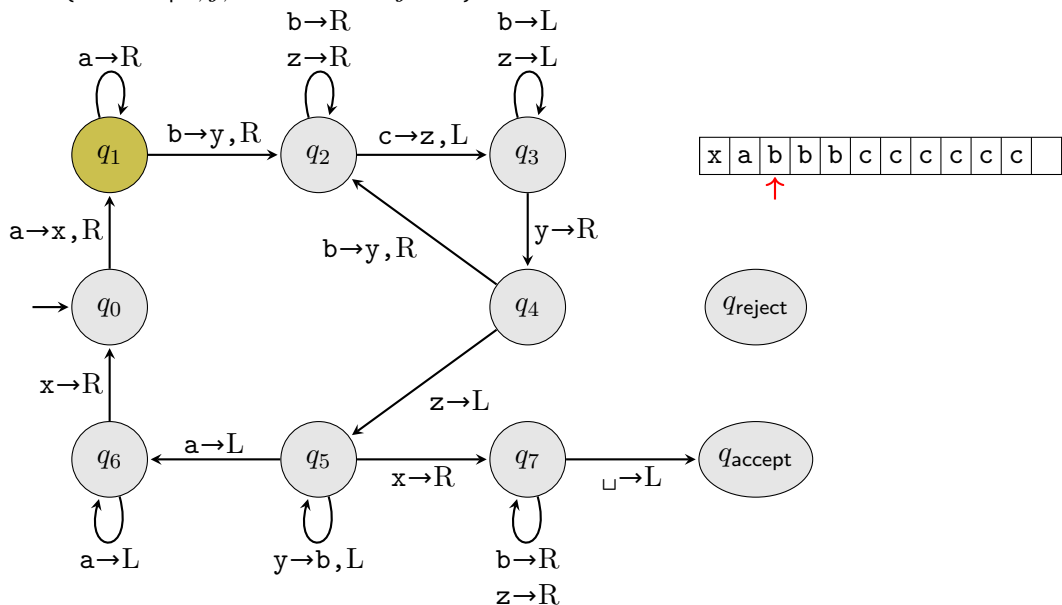
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# Multiplication TM

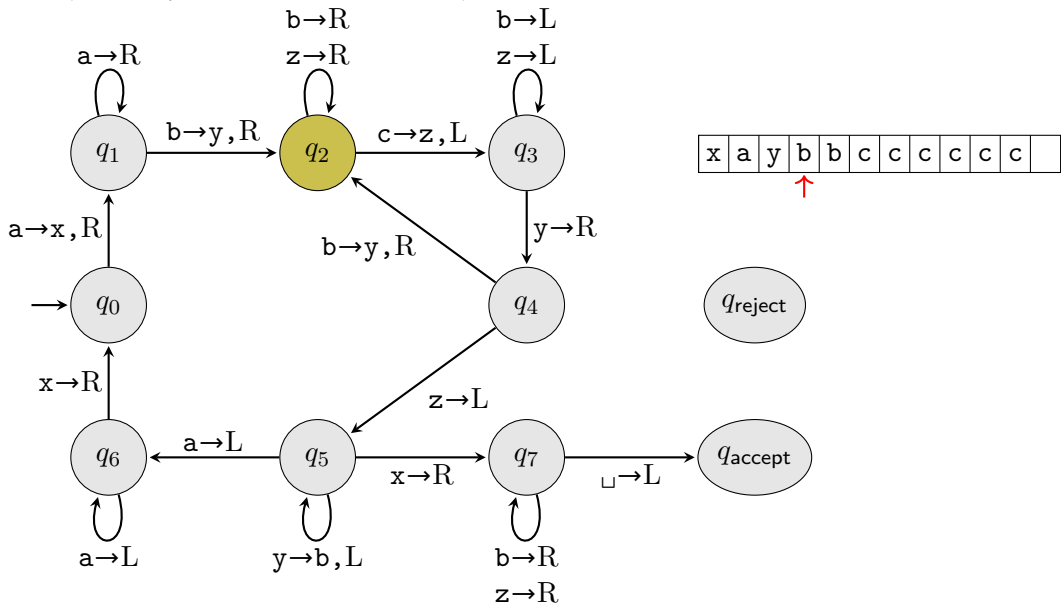
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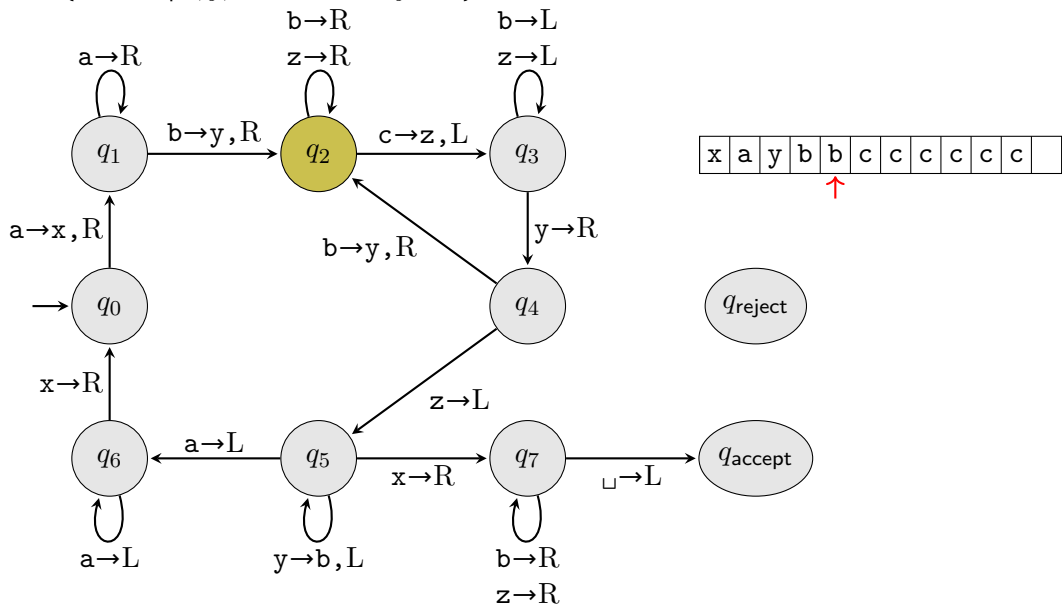
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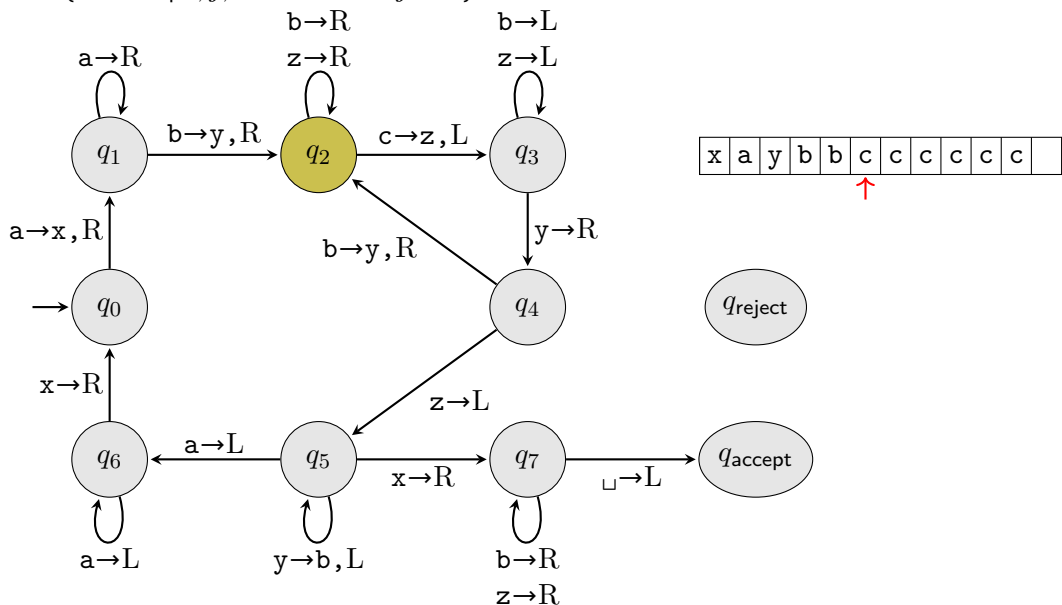
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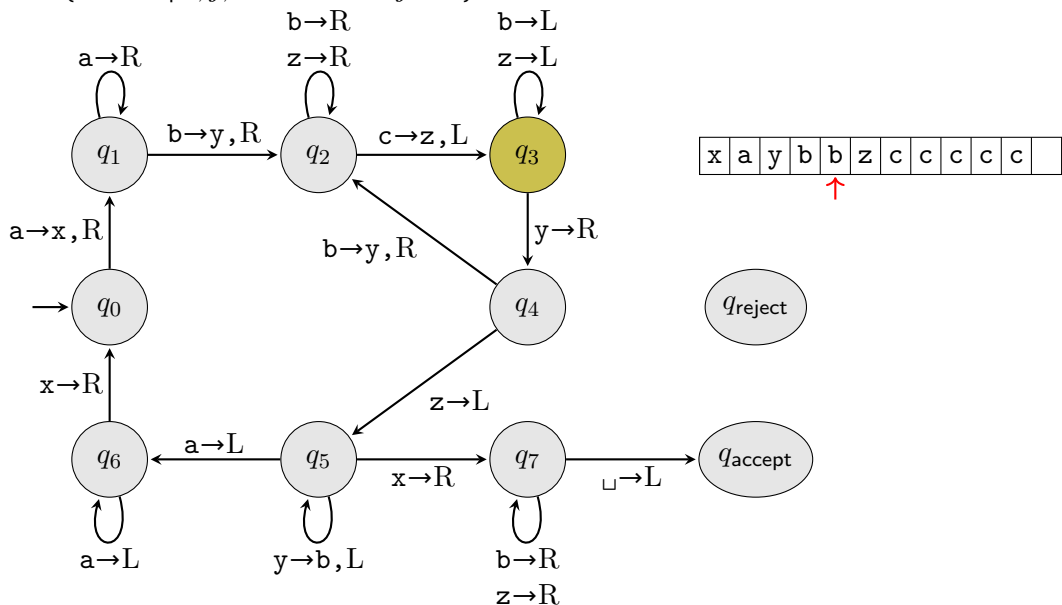
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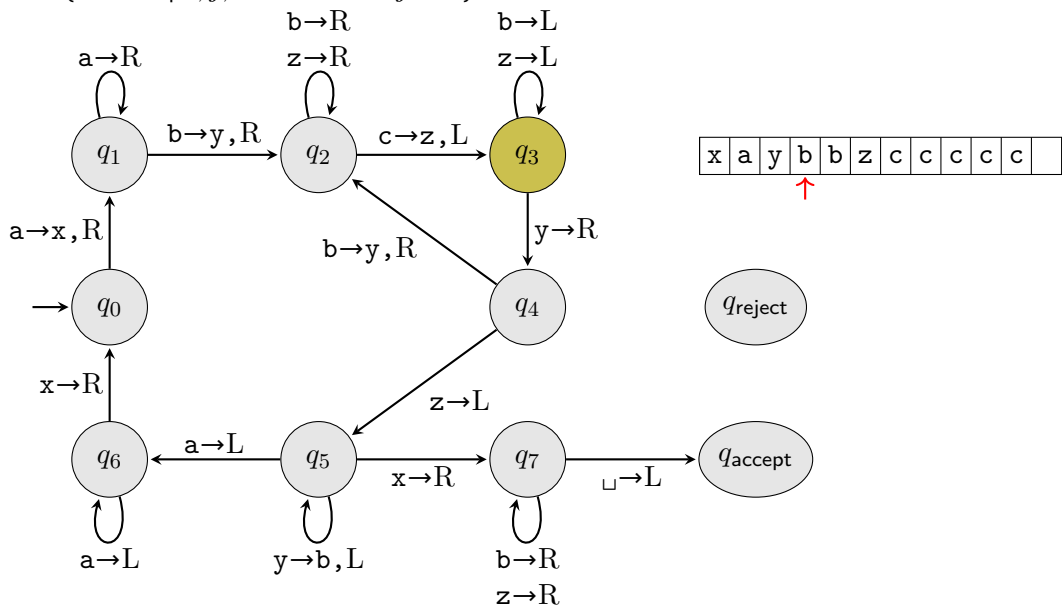
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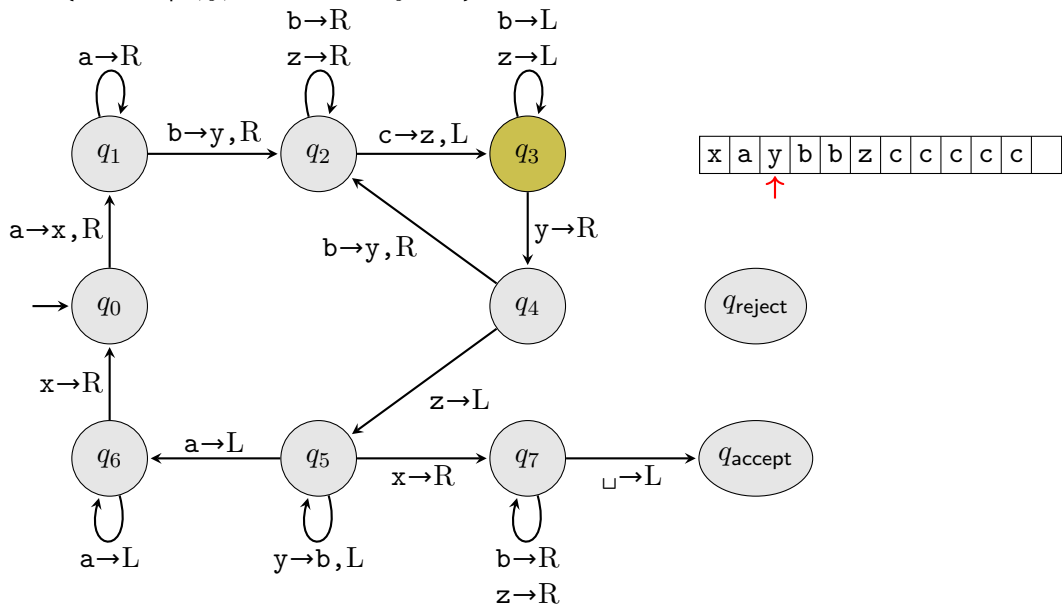
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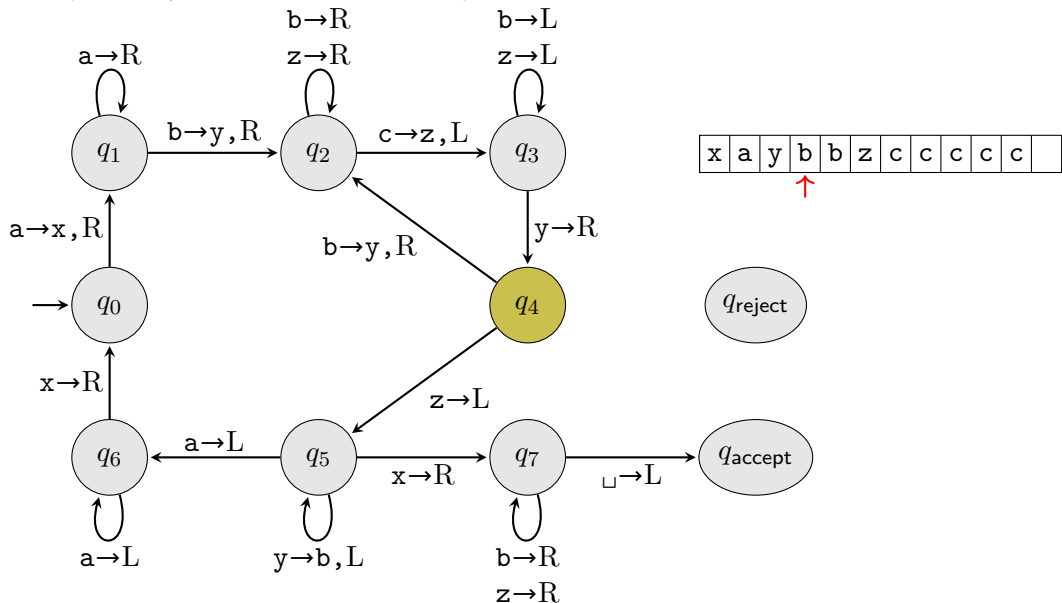
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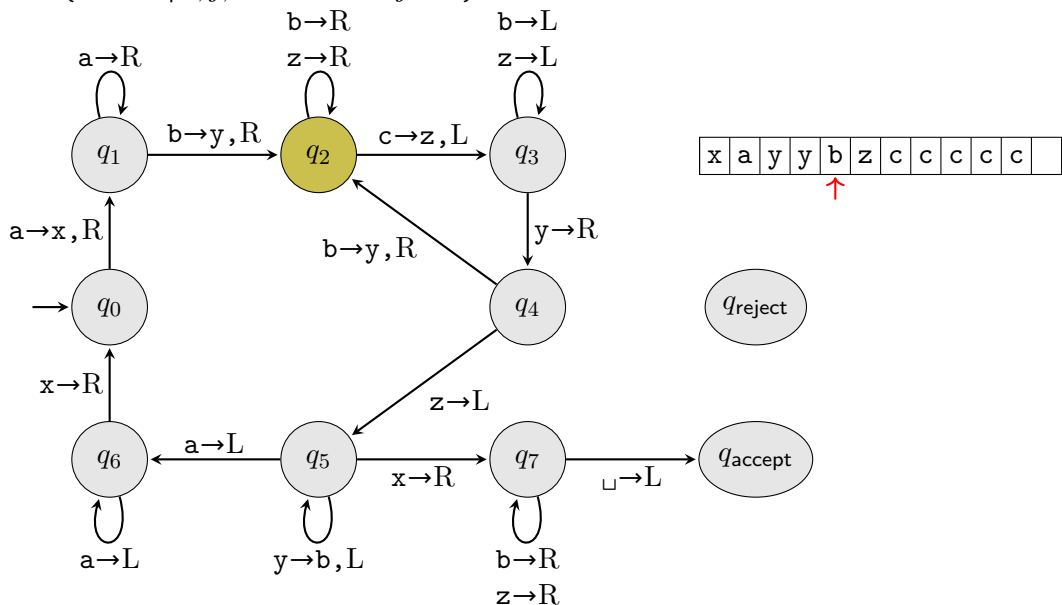
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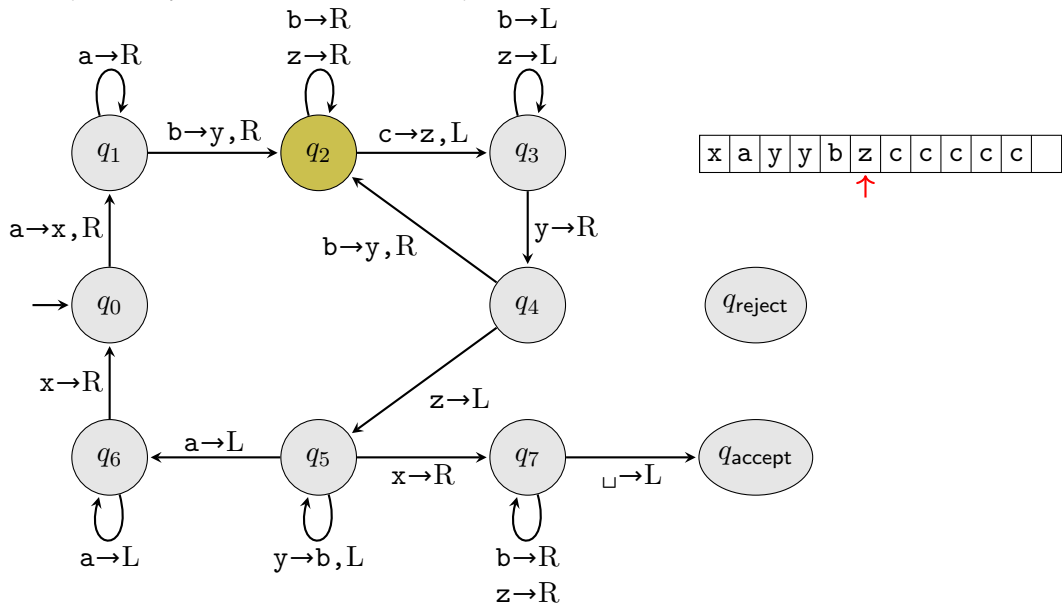
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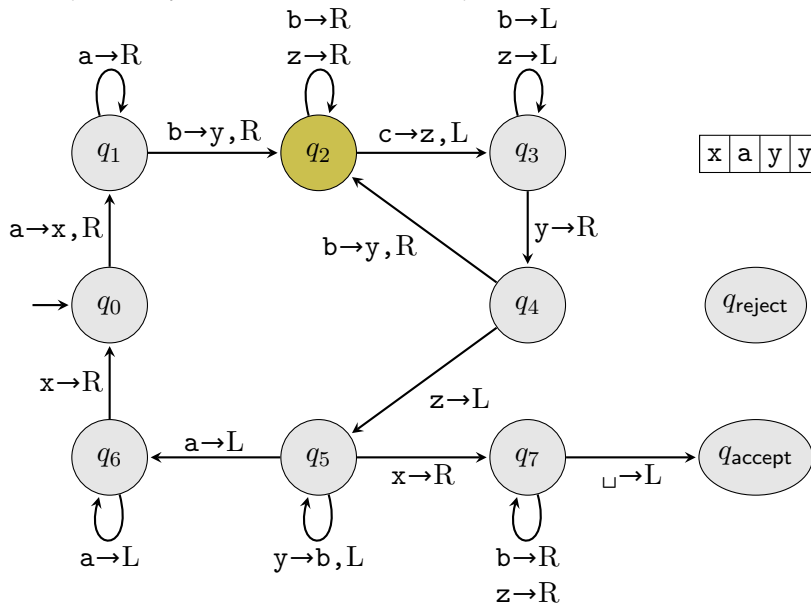
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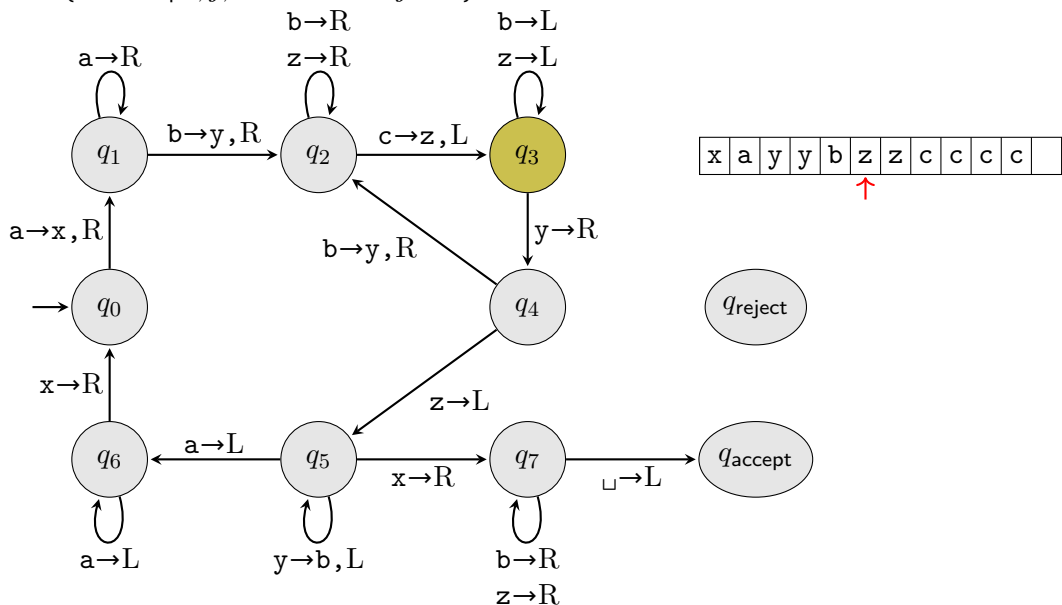
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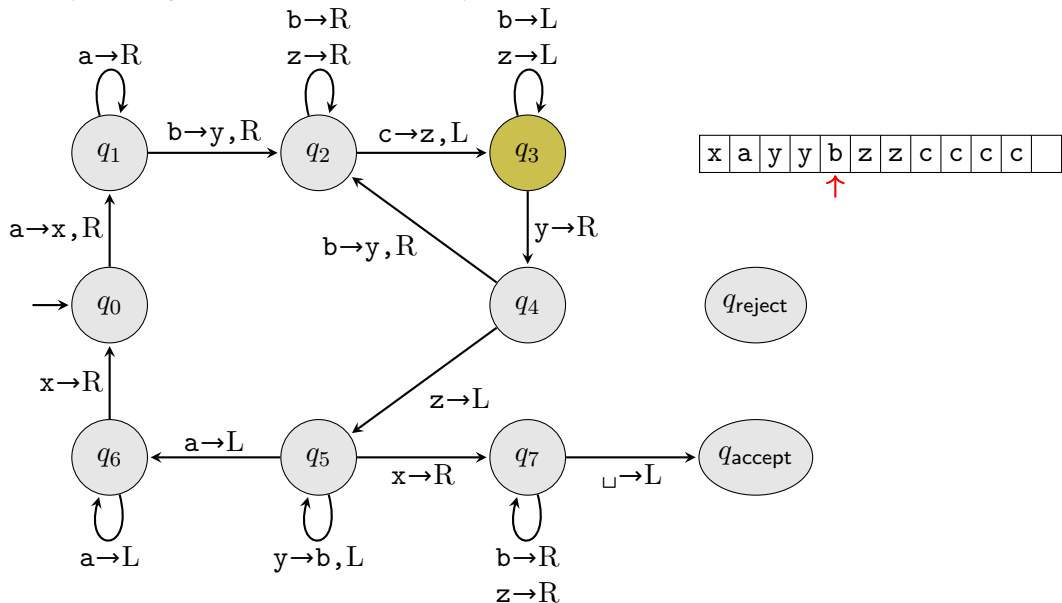
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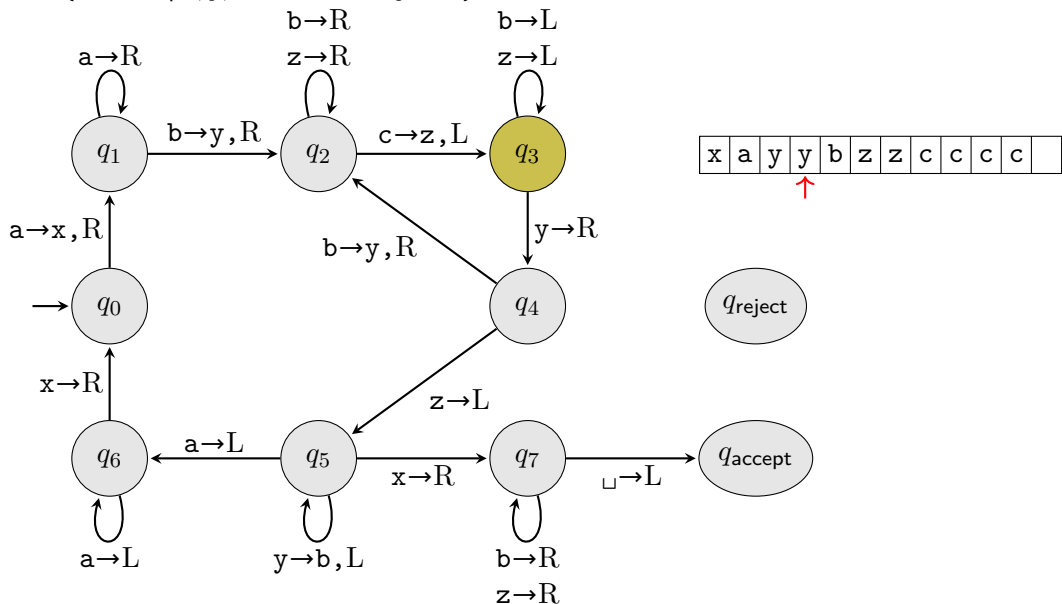
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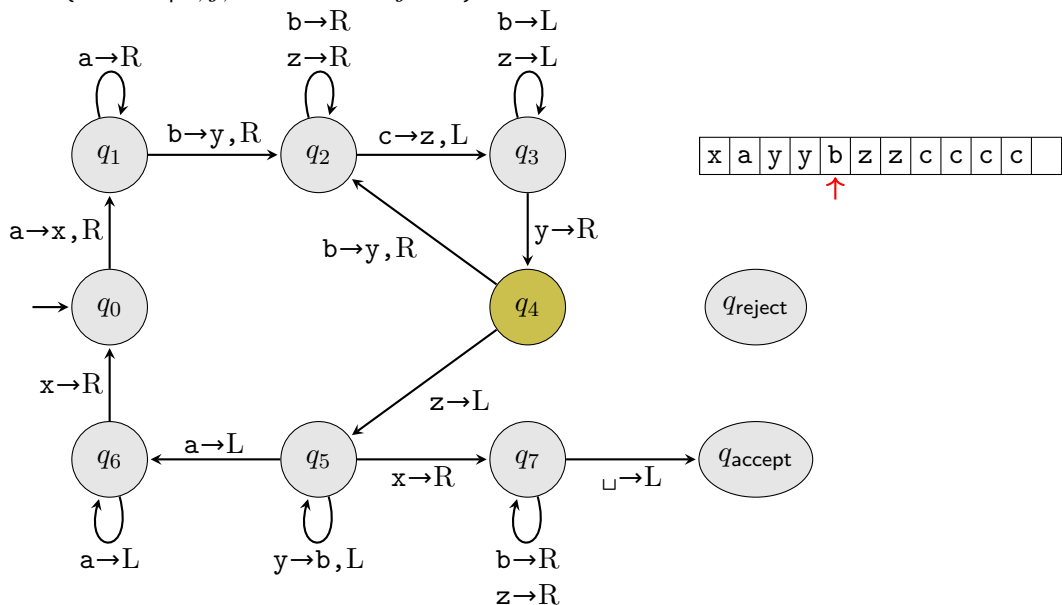
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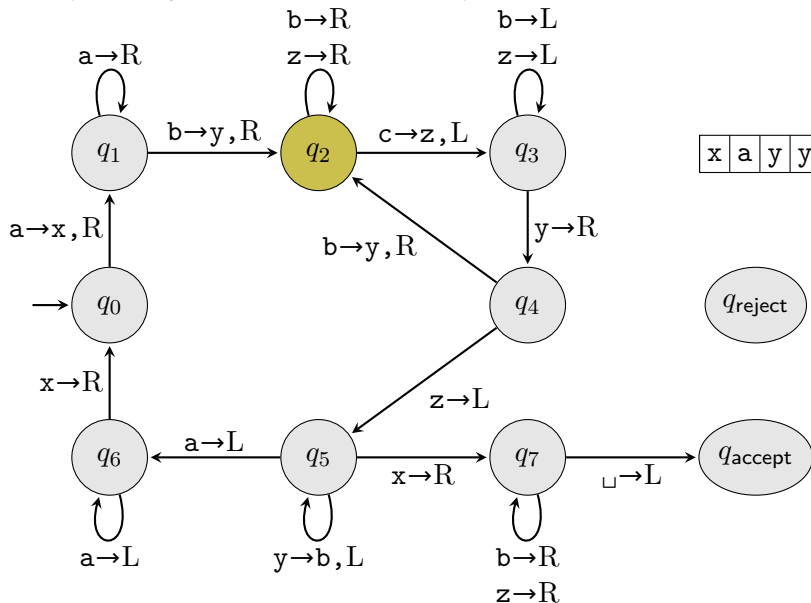
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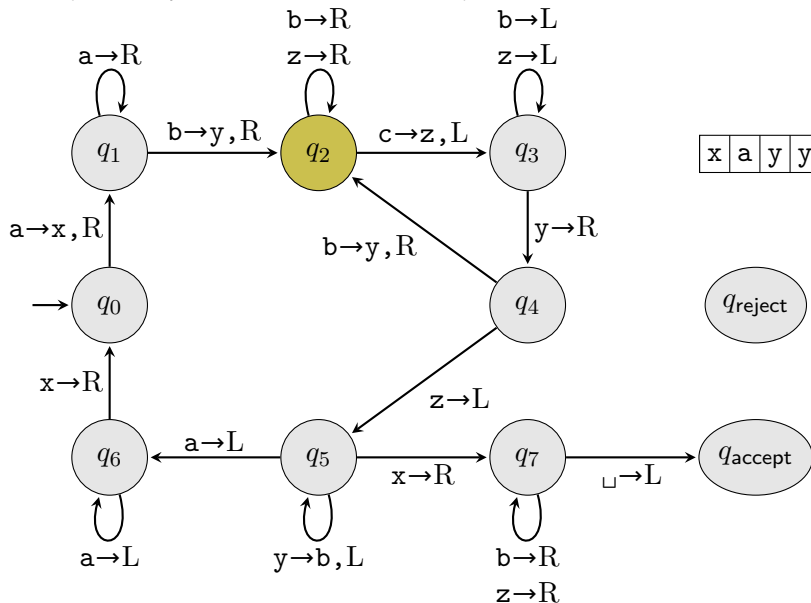
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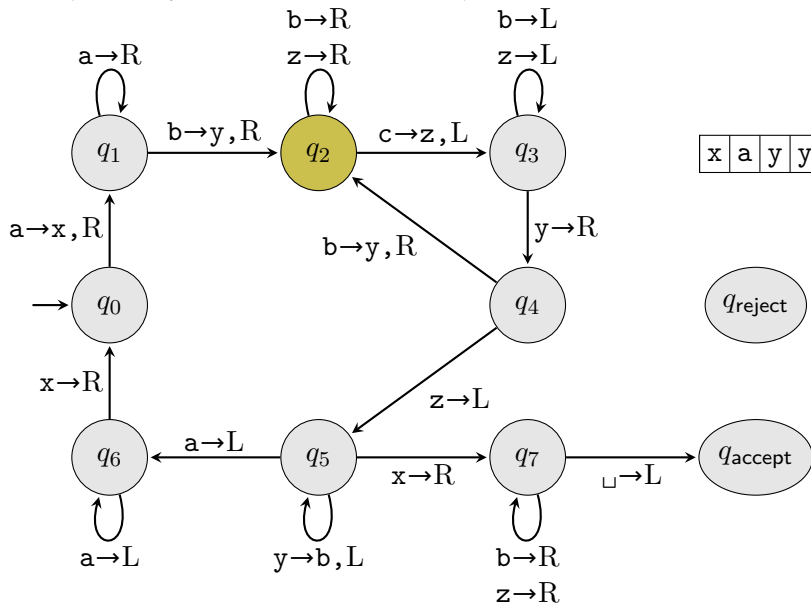
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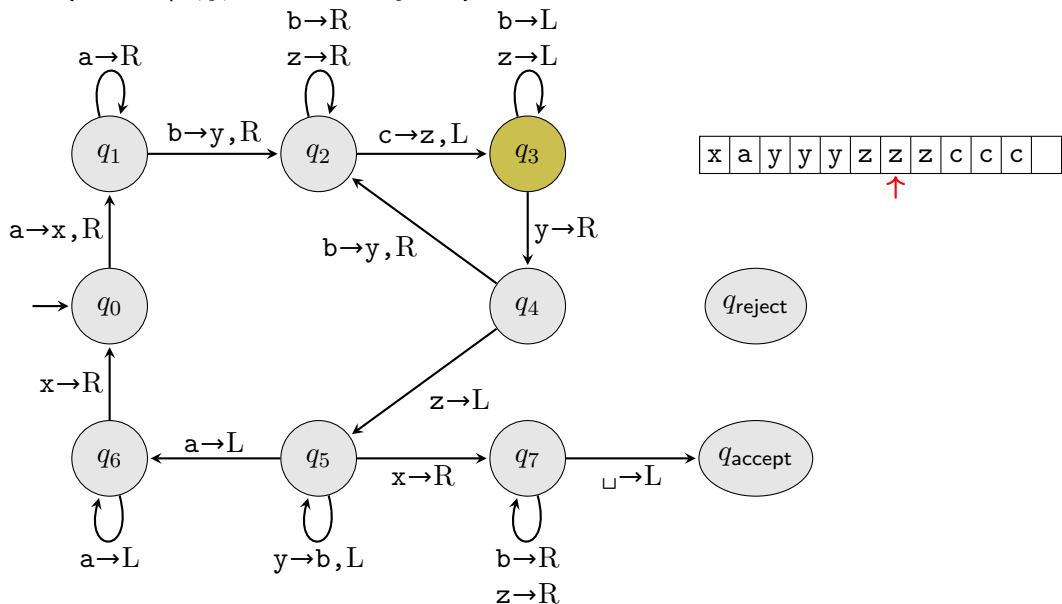
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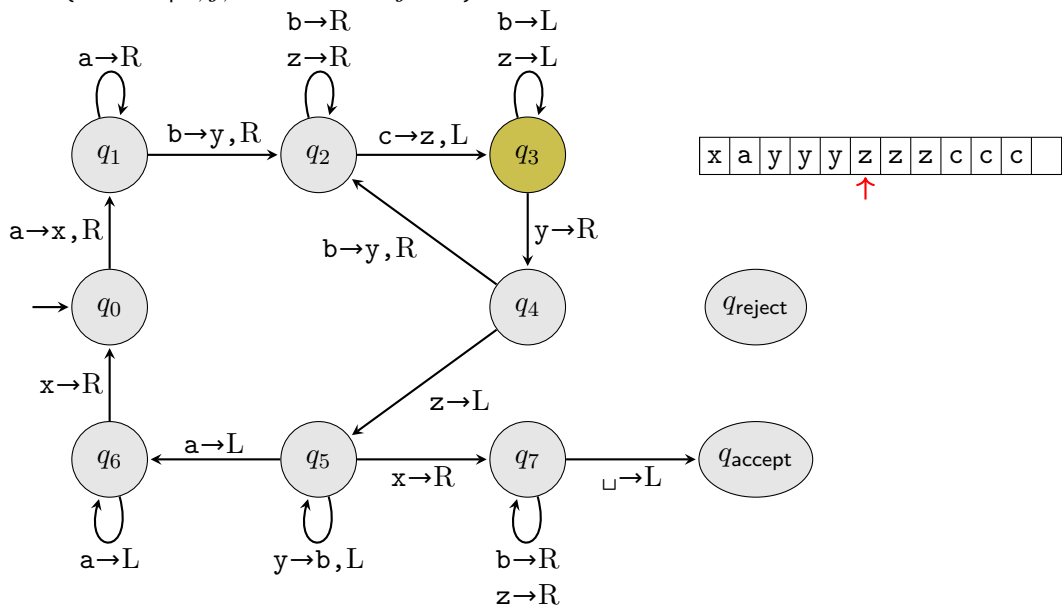
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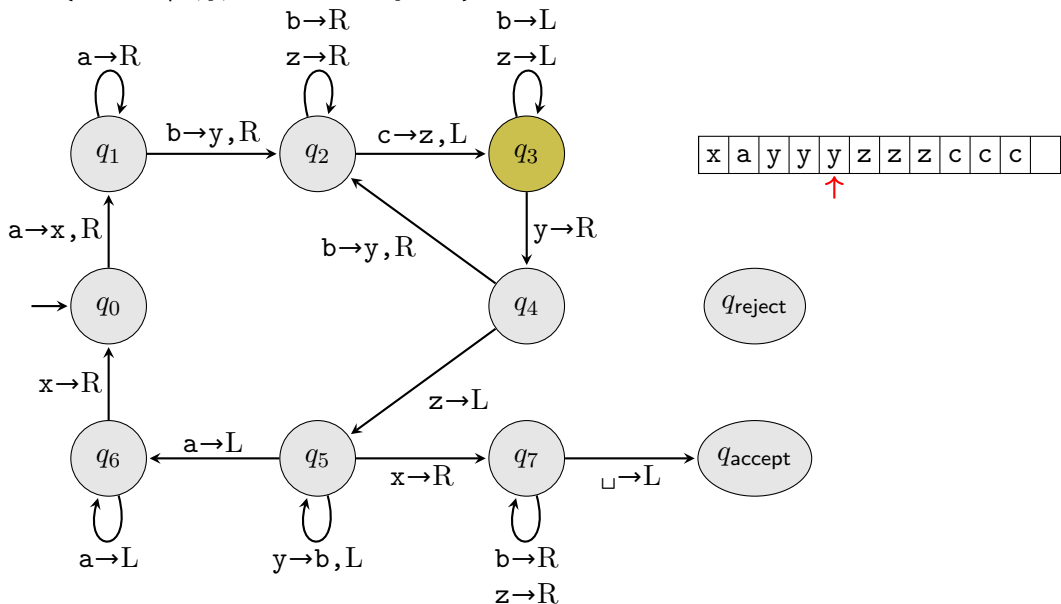
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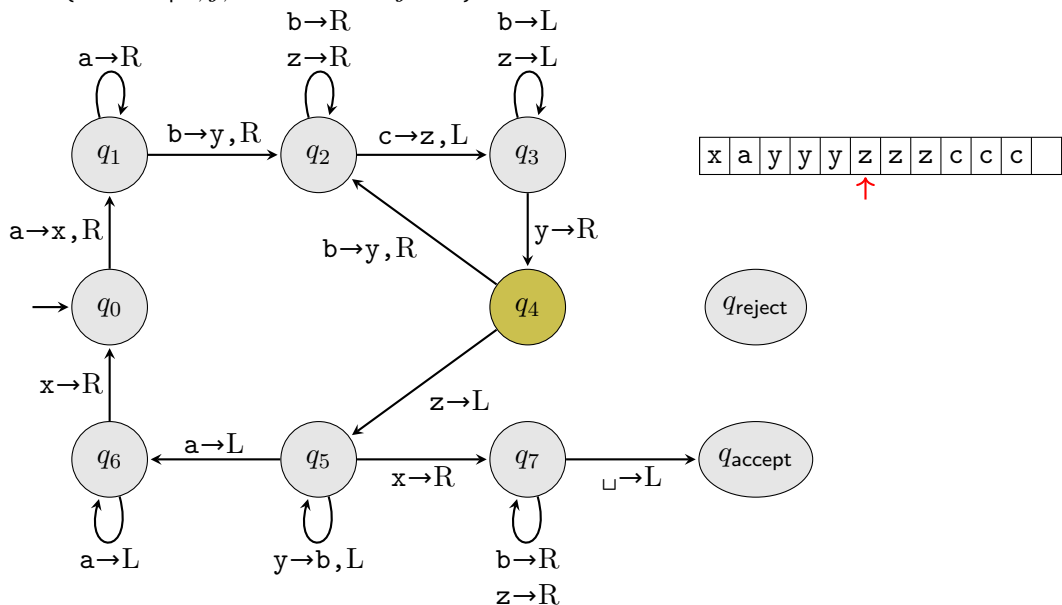
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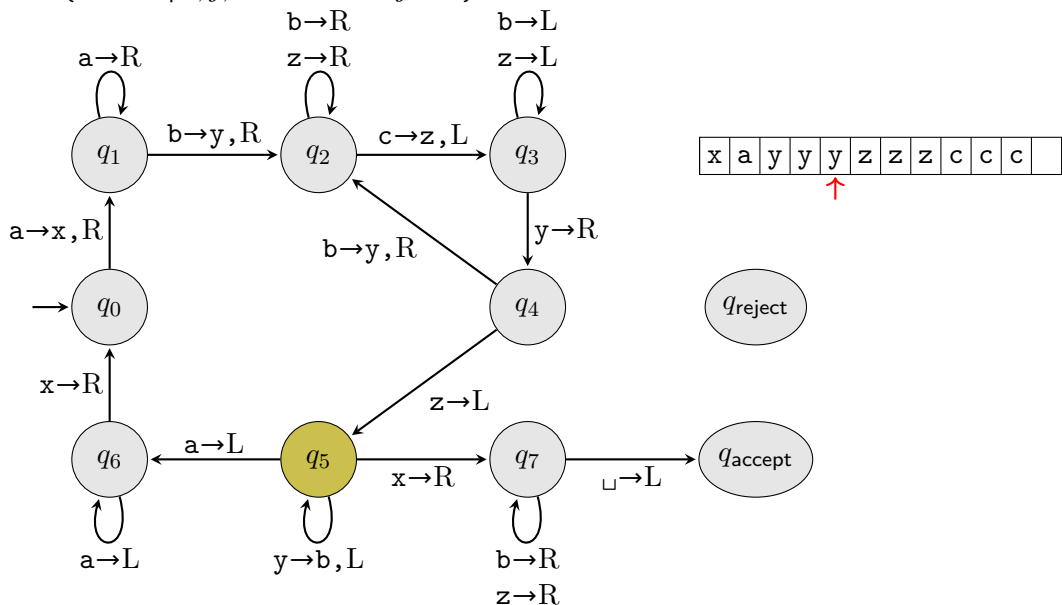
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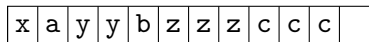
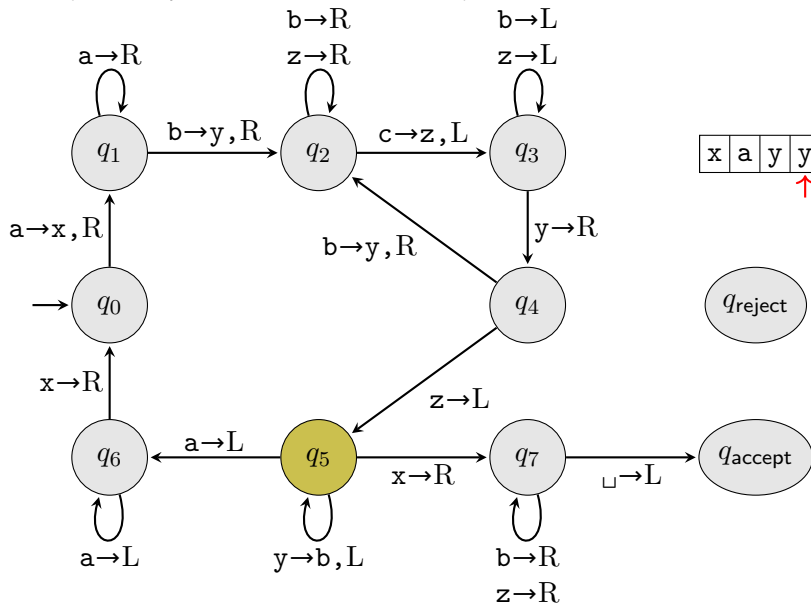
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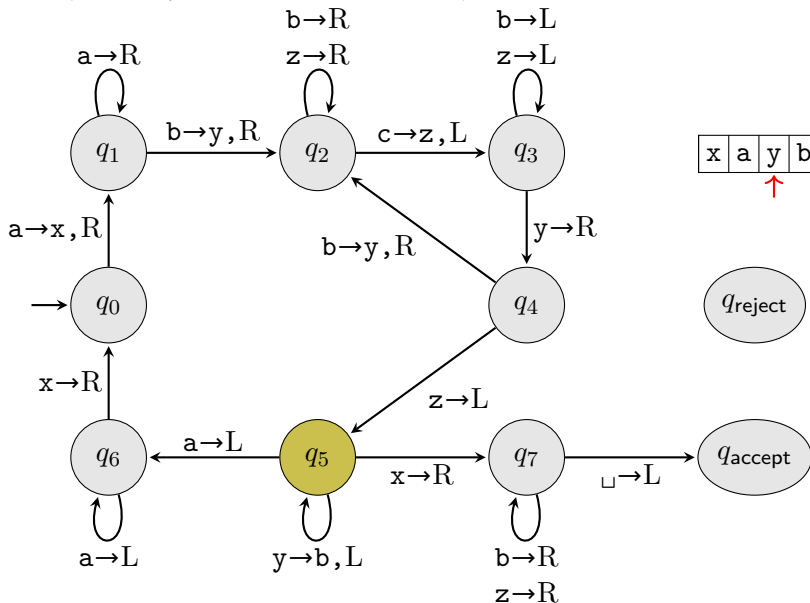
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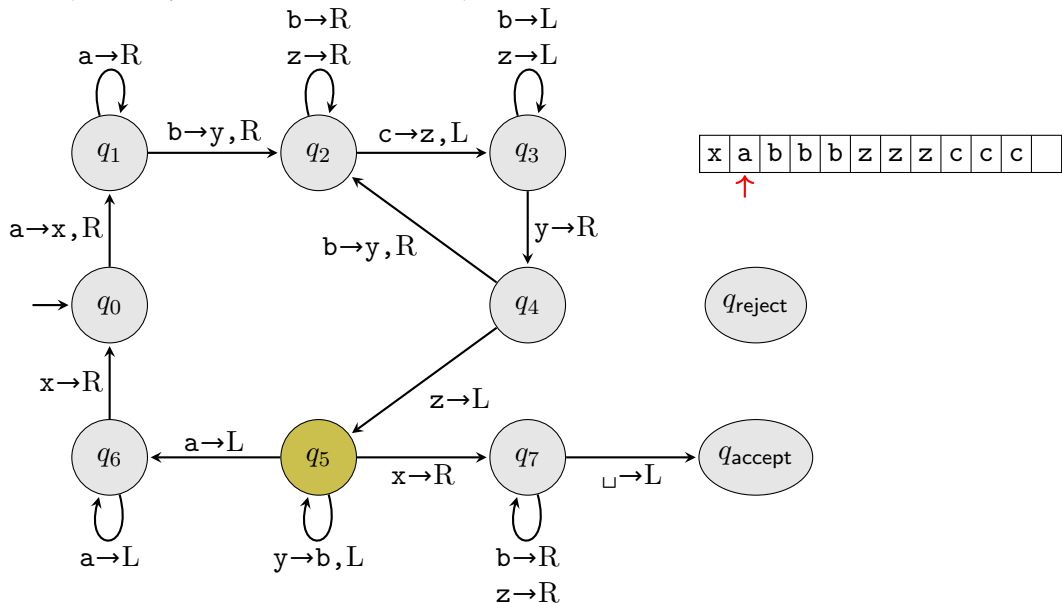
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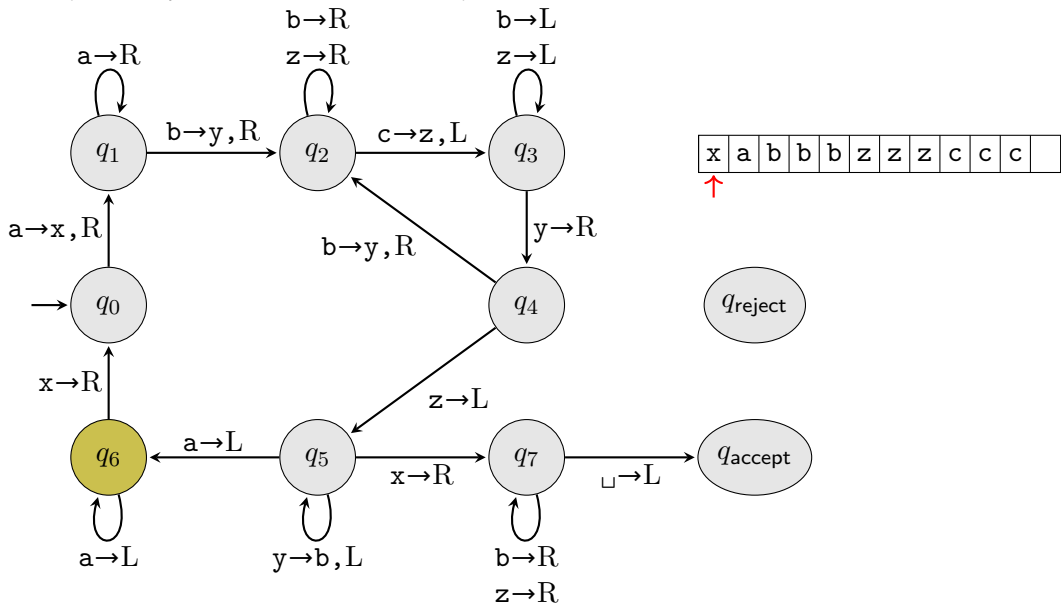
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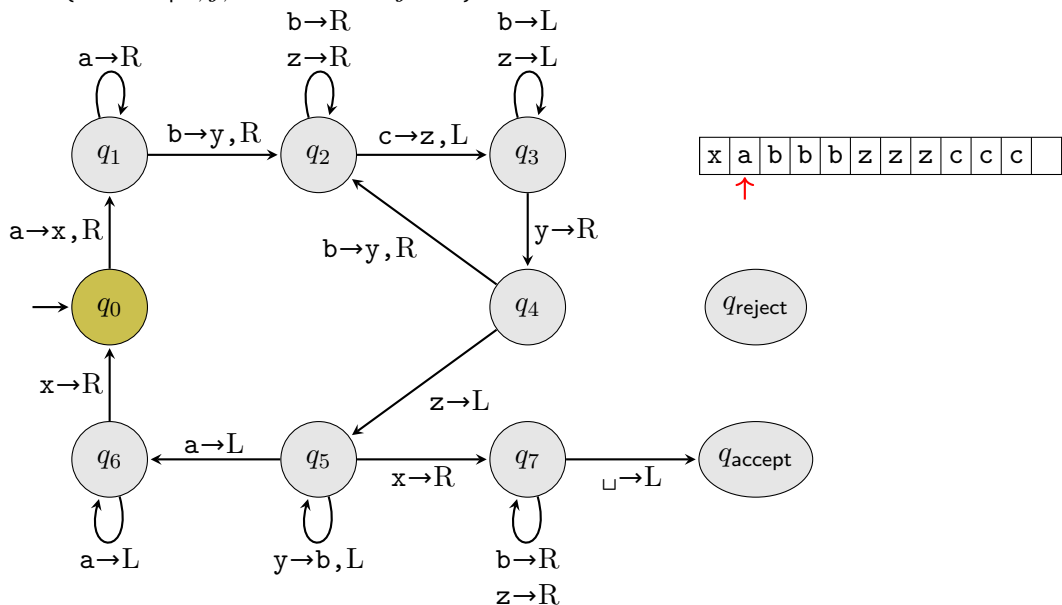
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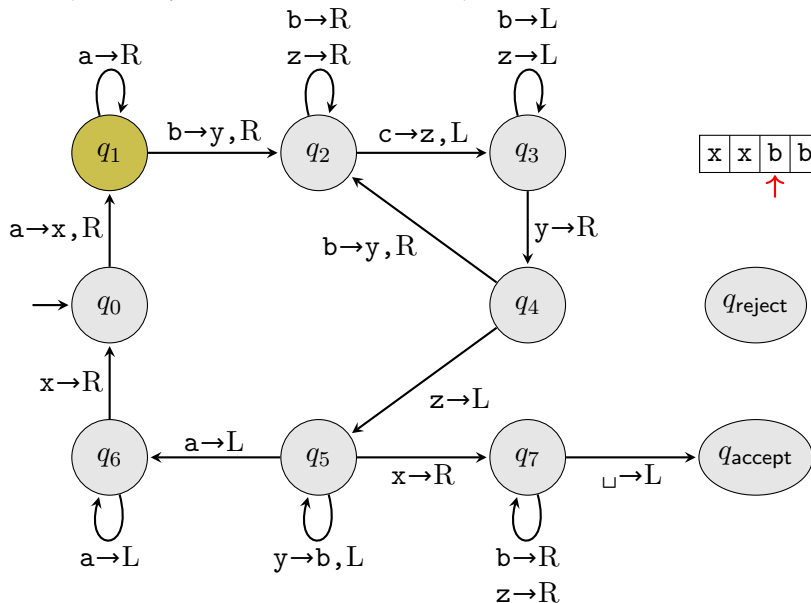
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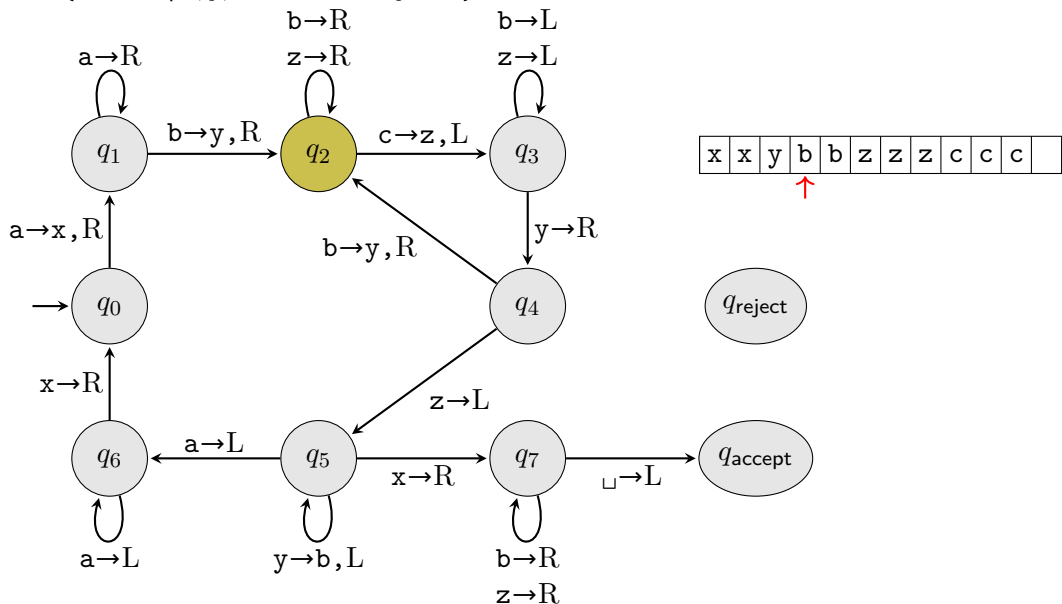
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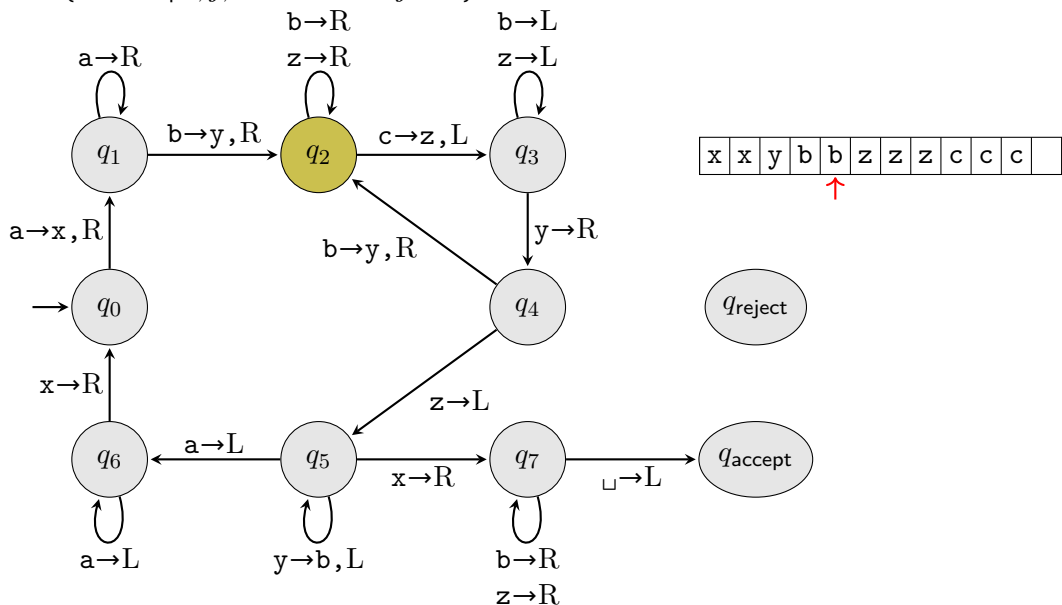
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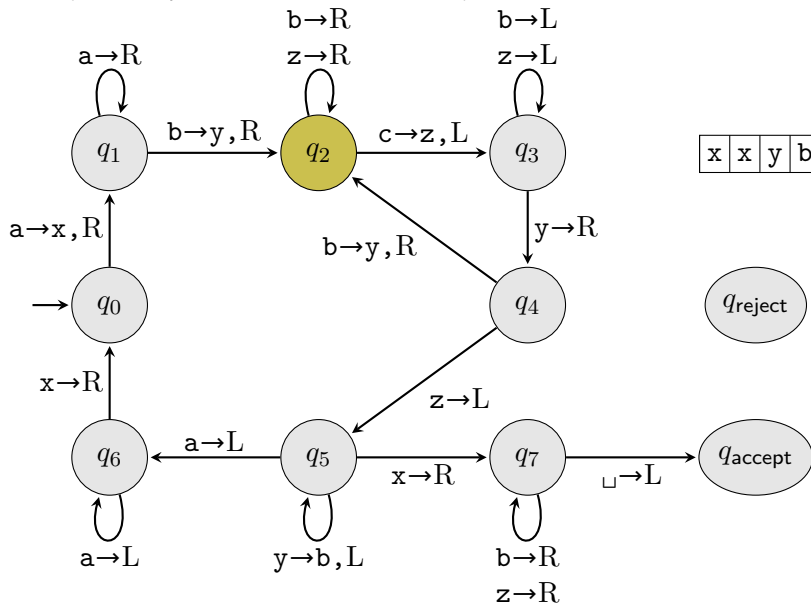
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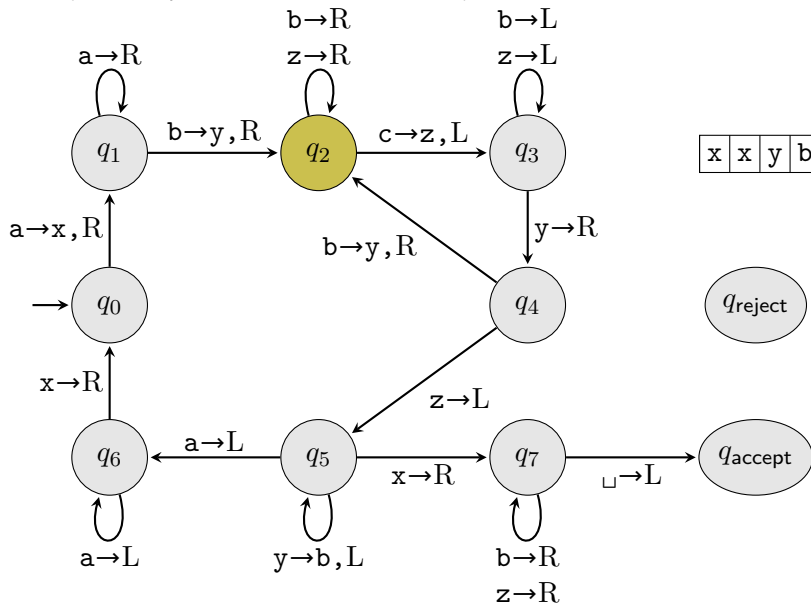
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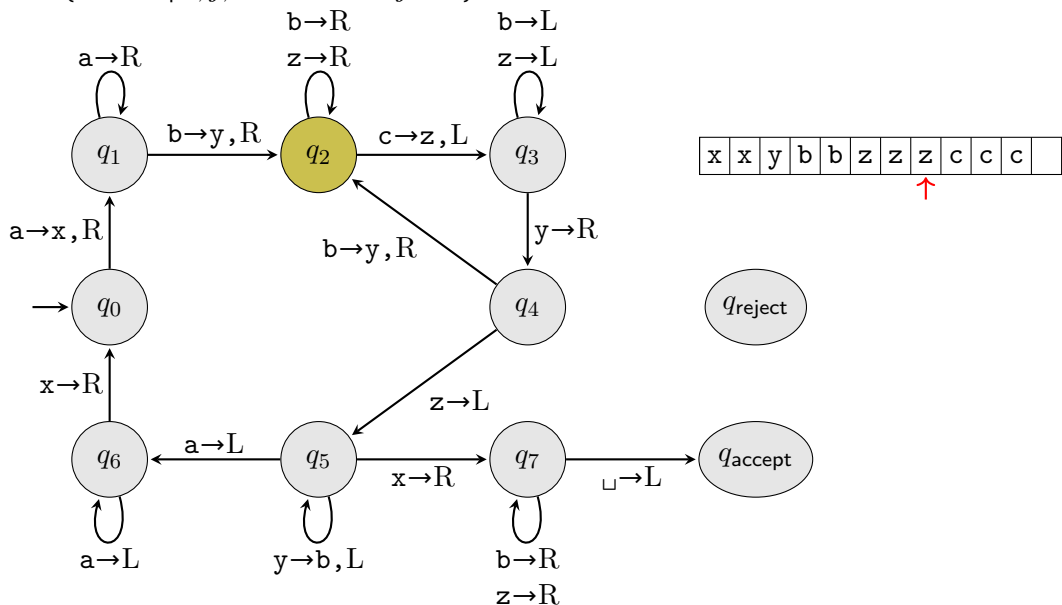
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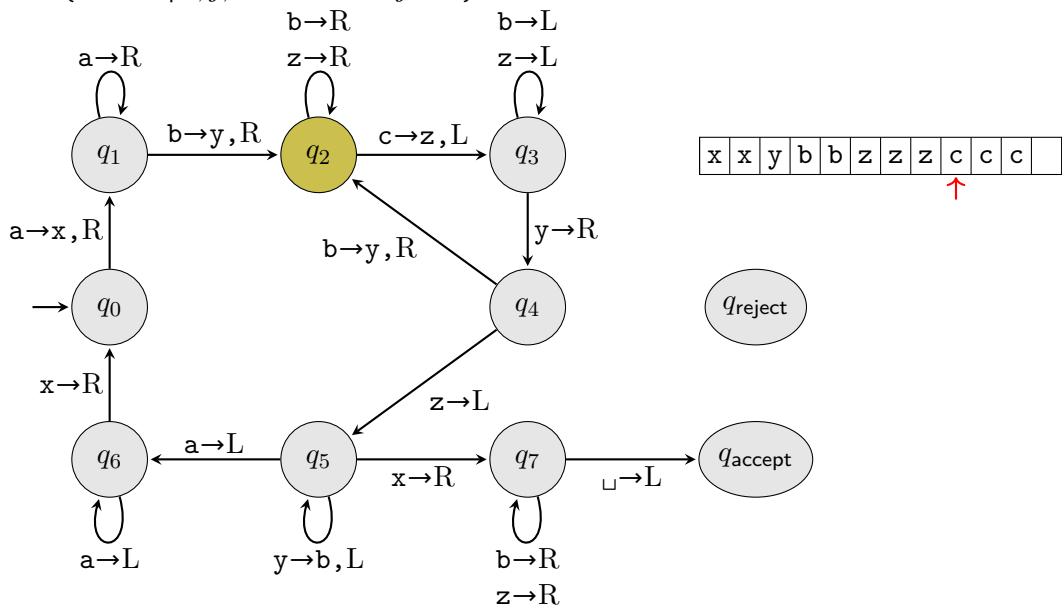
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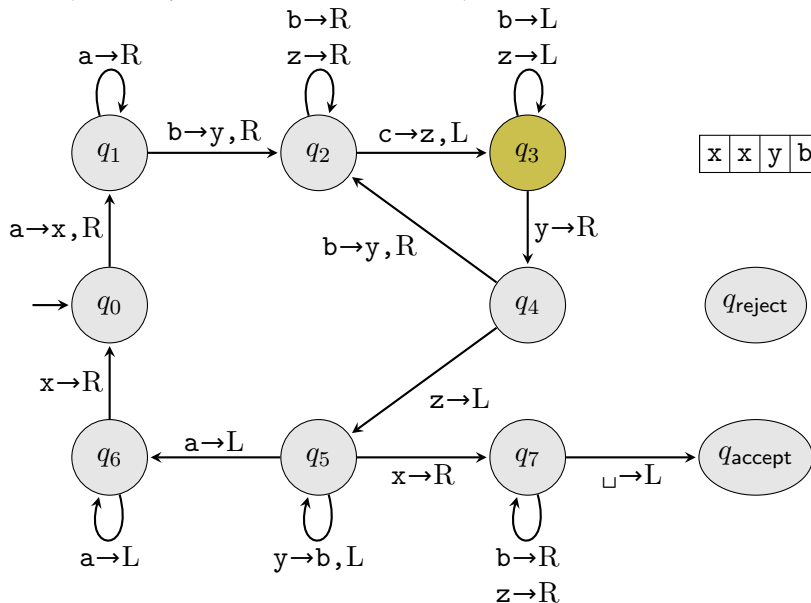
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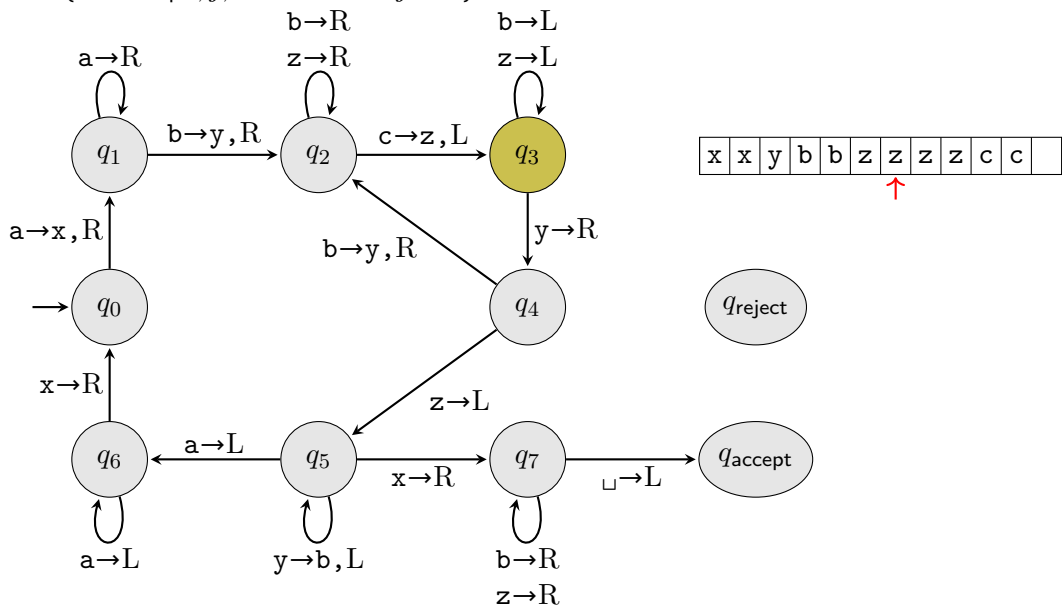
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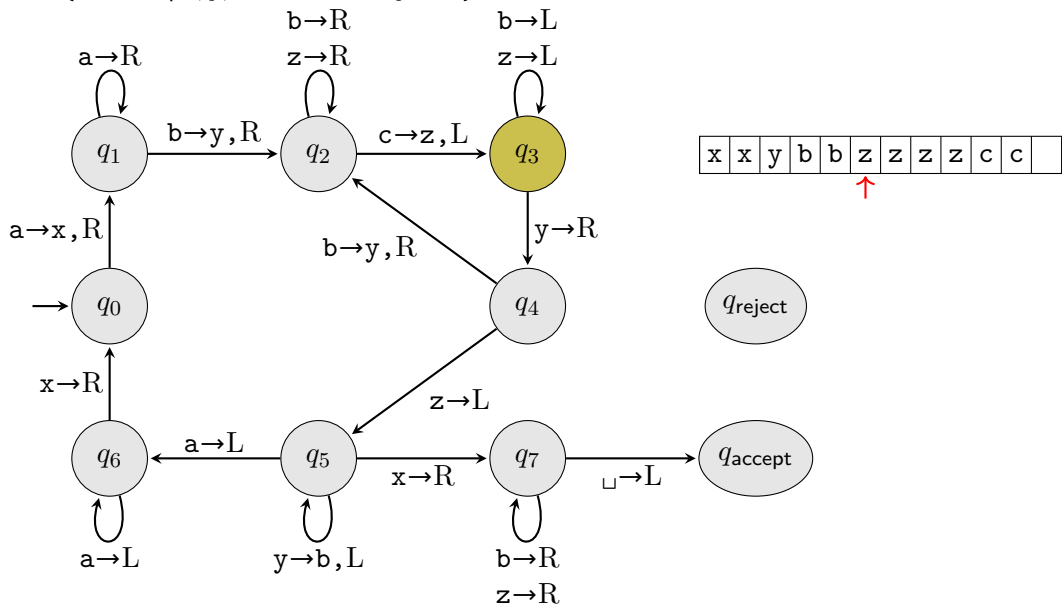
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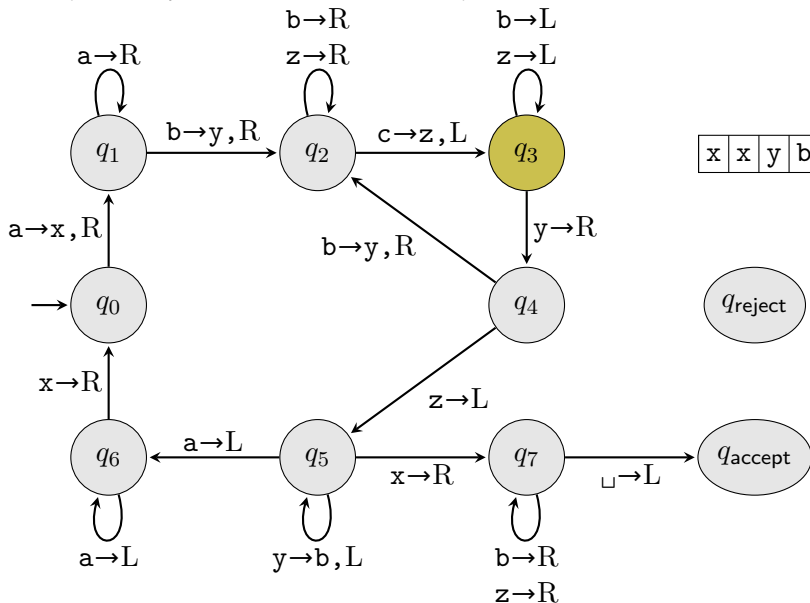
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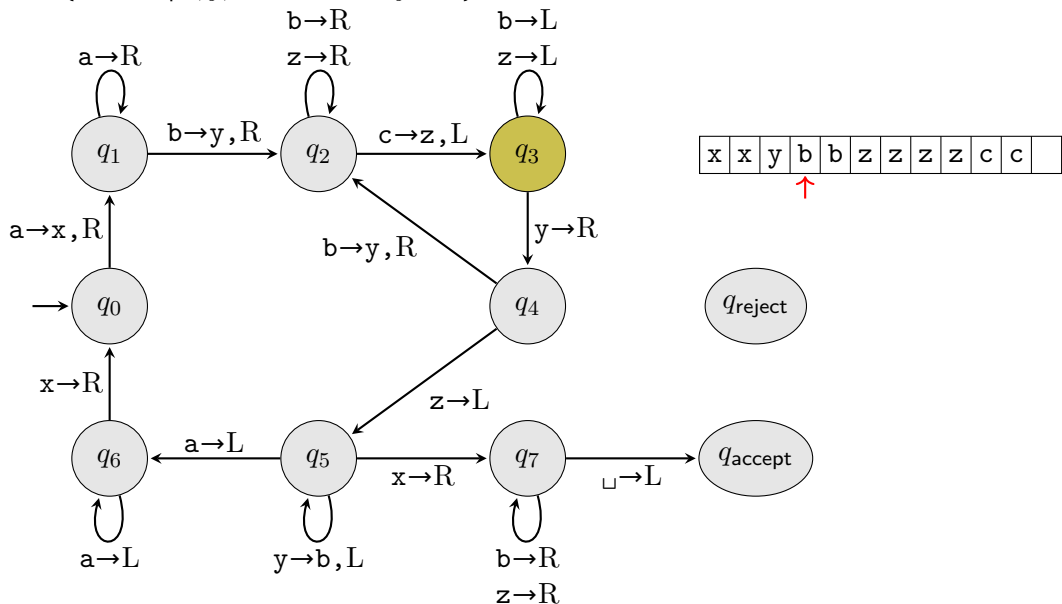
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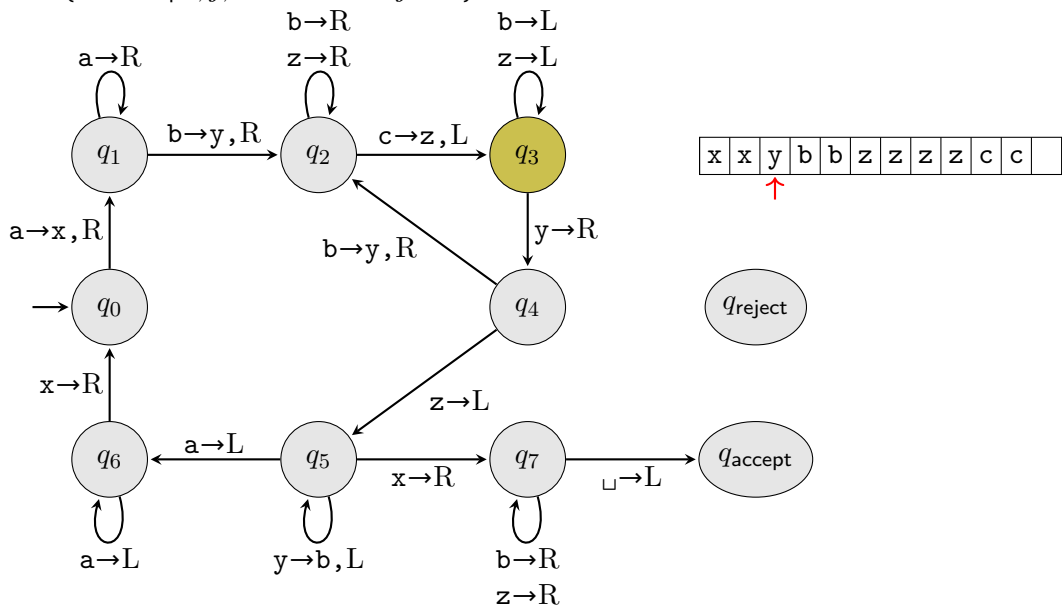
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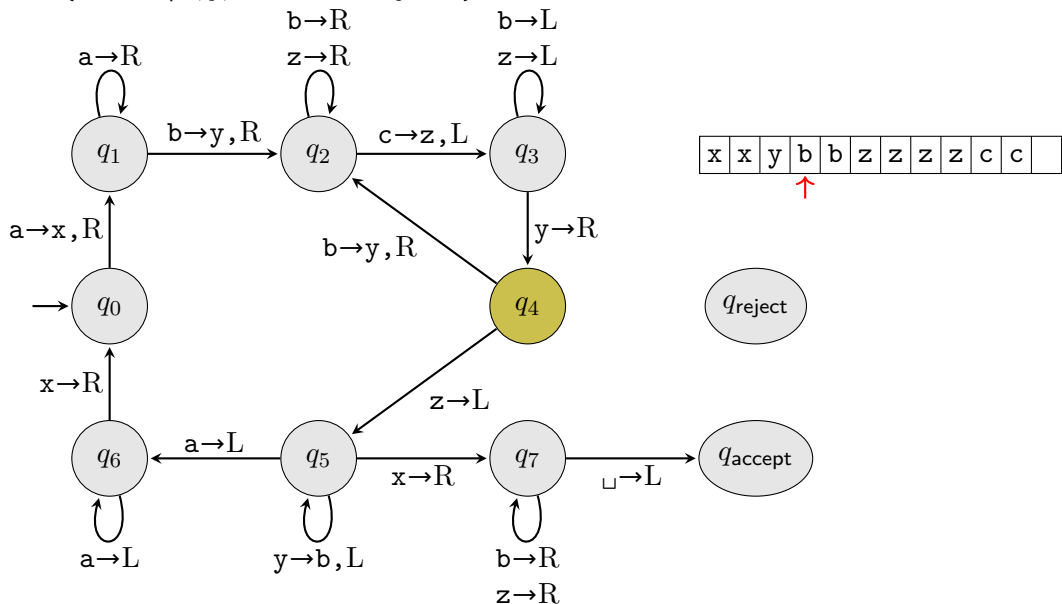
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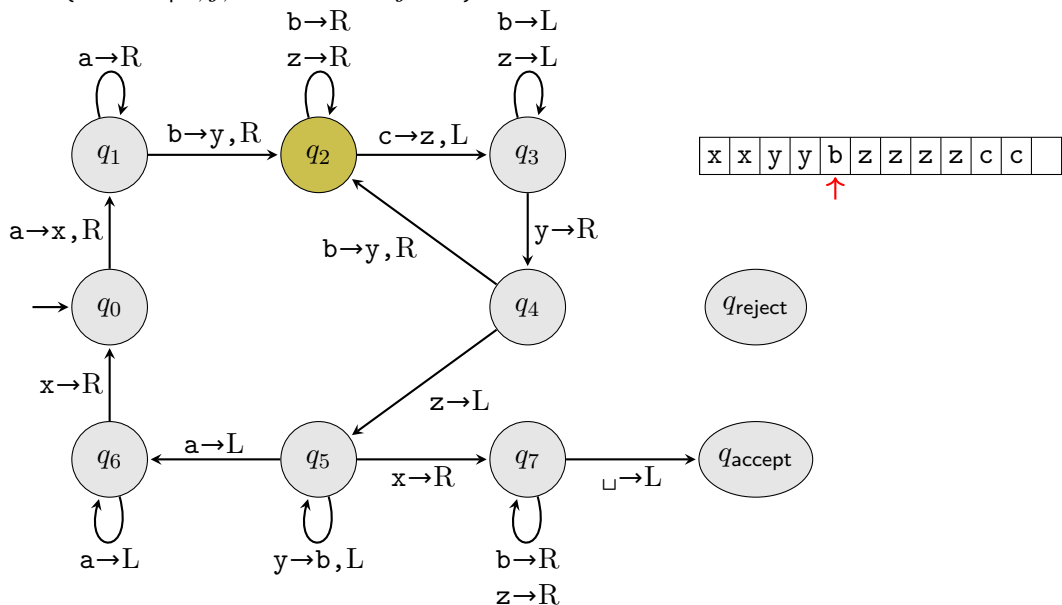
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$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = \text{aabbbcccccc}$



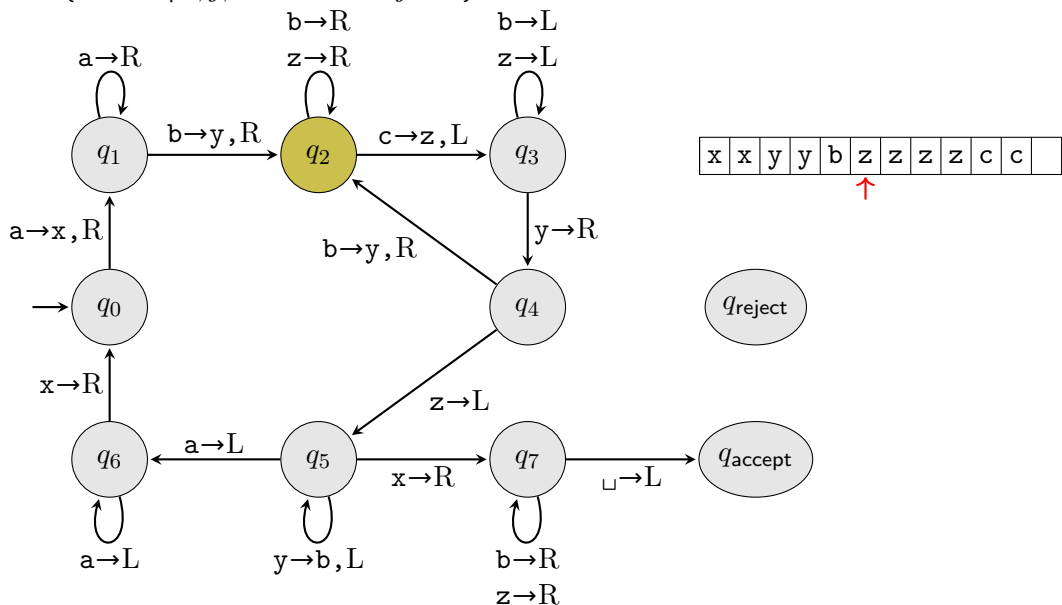
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$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = aabbcccccc$



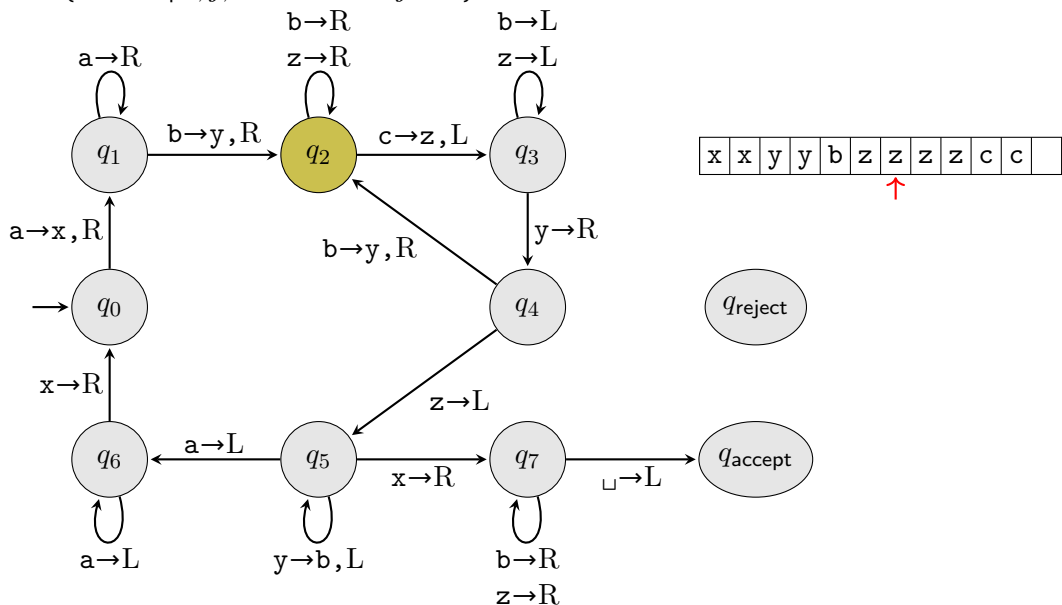
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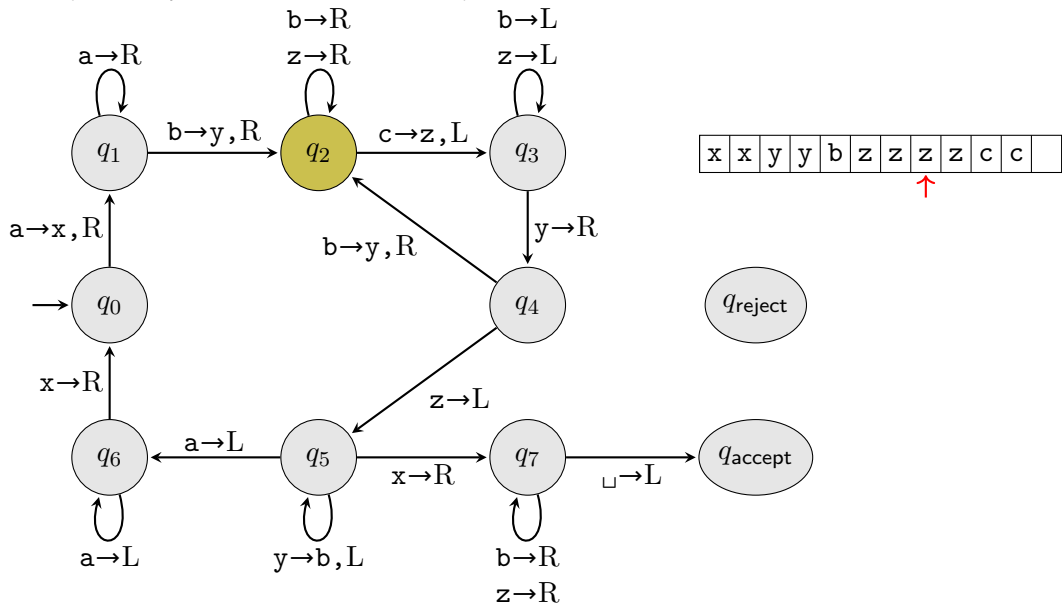
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$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = \text{aabbbcccccc}$



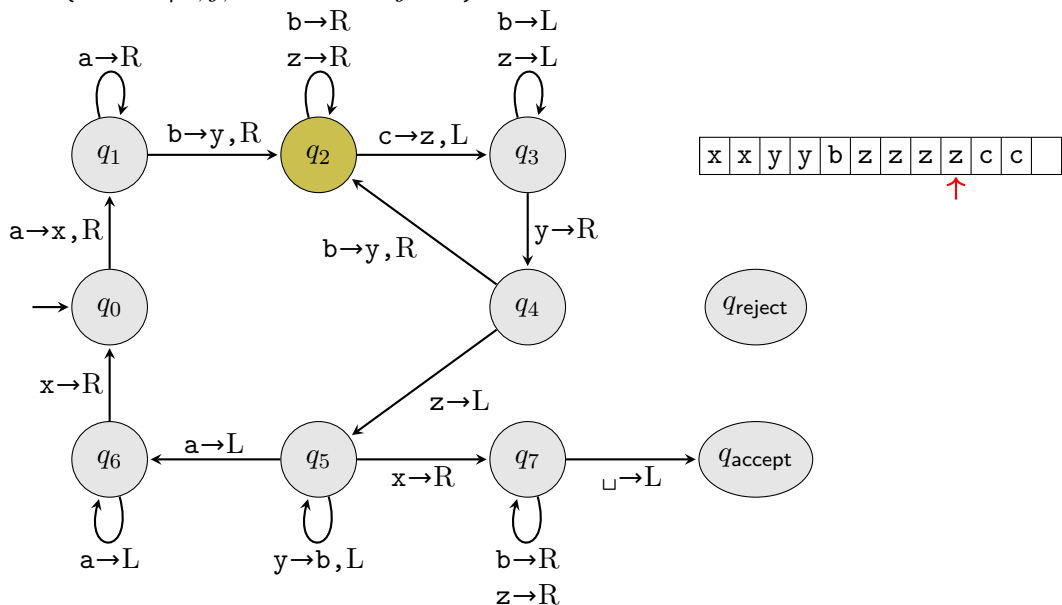
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$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = \text{aabbbcccccc}$



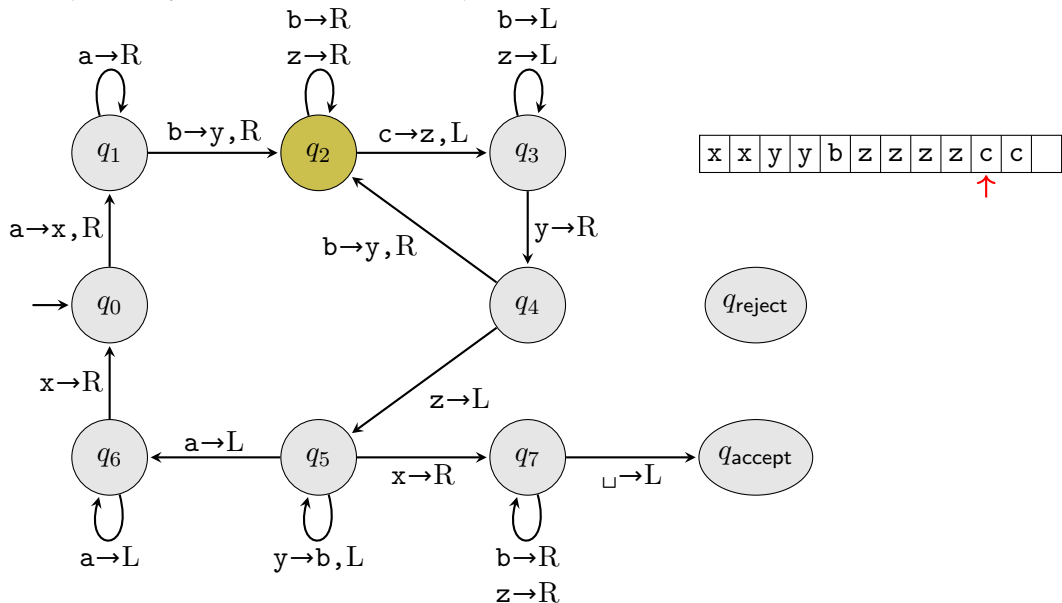
# Multiplication TM

$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = \text{aabbbcccccc}$



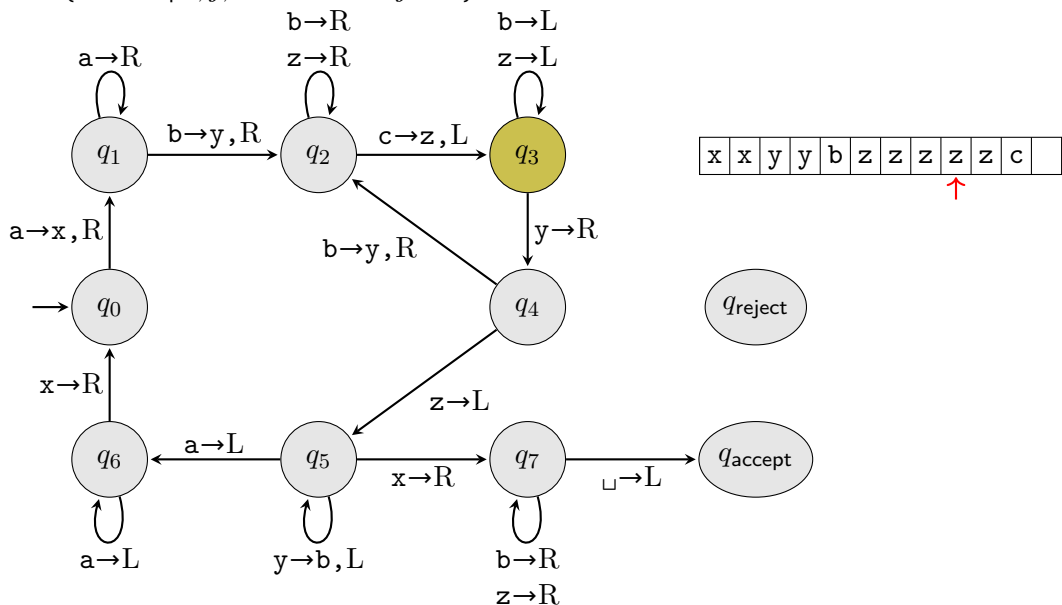
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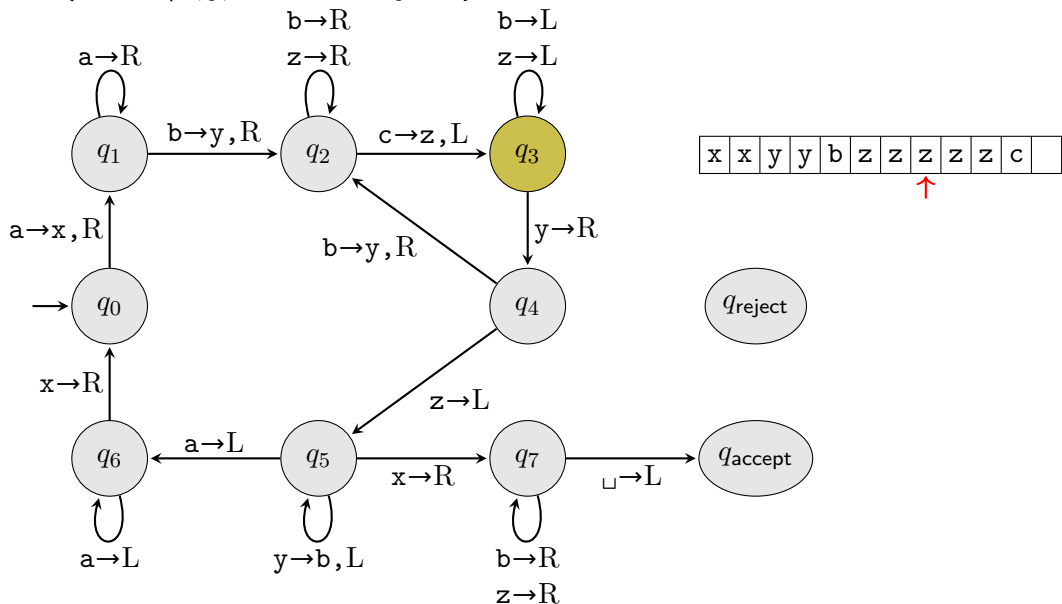
$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = \text{aabbcccccc}$





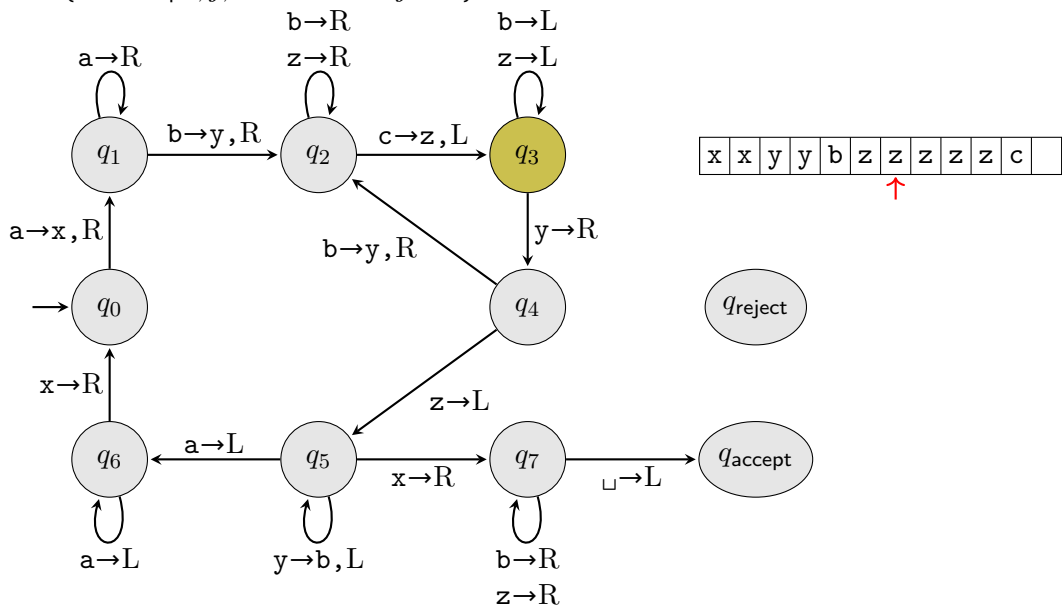
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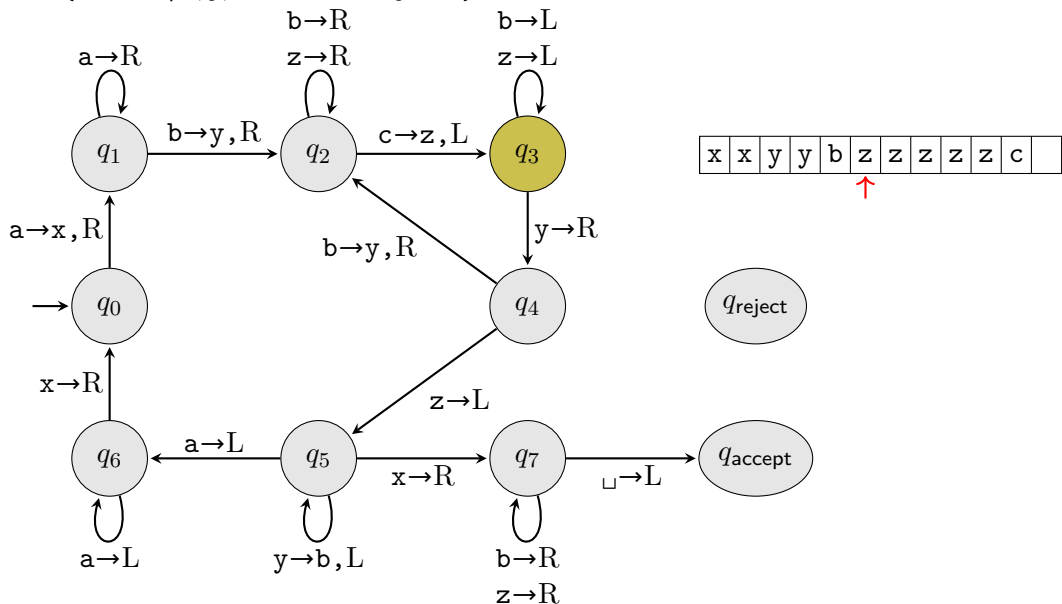
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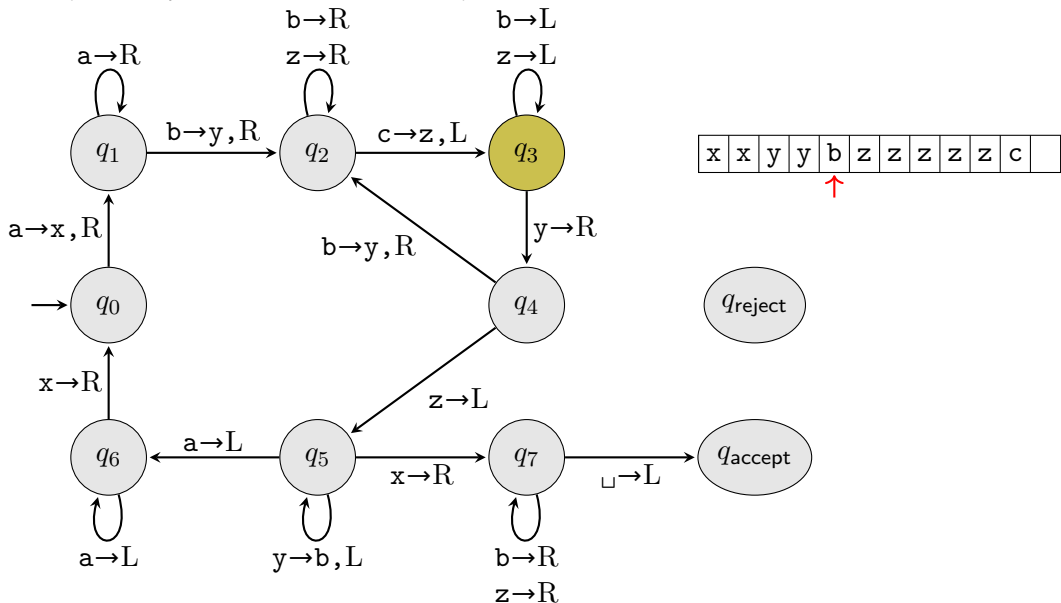
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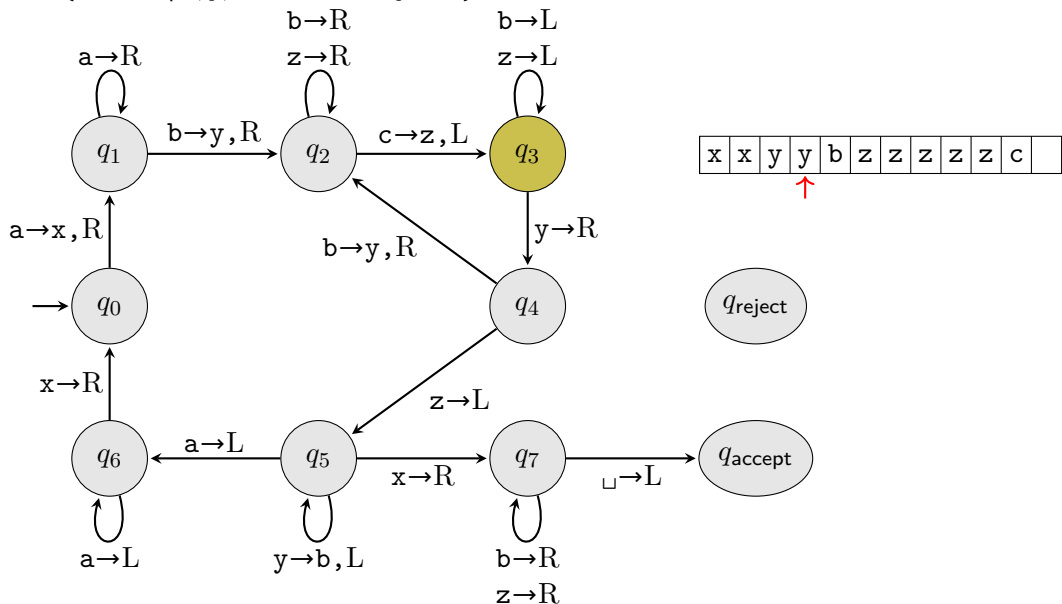
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$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = \text{aabbbcccccc}$



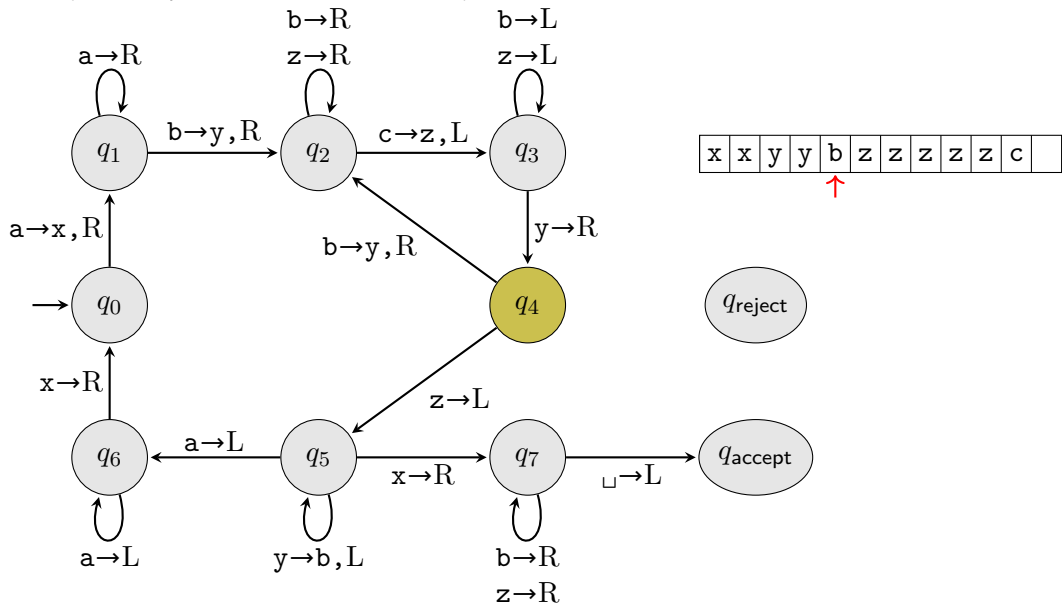
# Multiplication TM

$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = aabbcccccc$



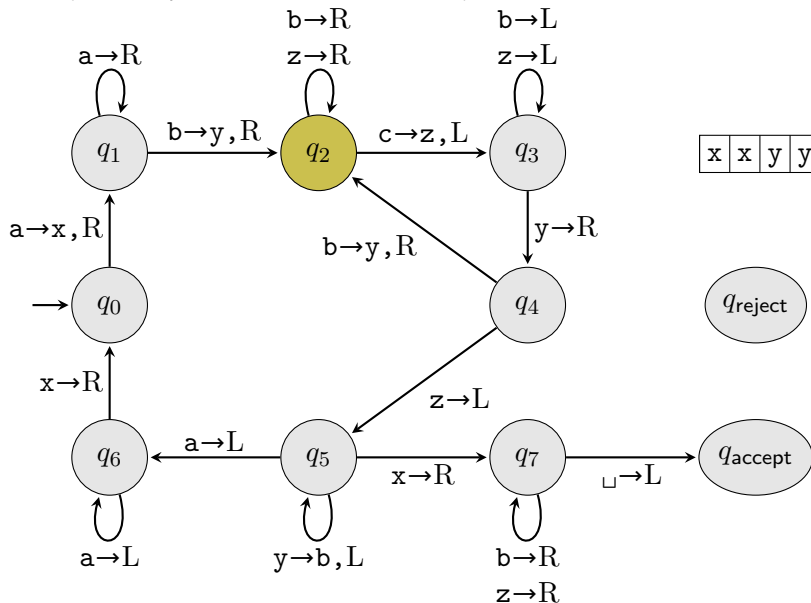
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$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = \text{aabbbcccccc}$



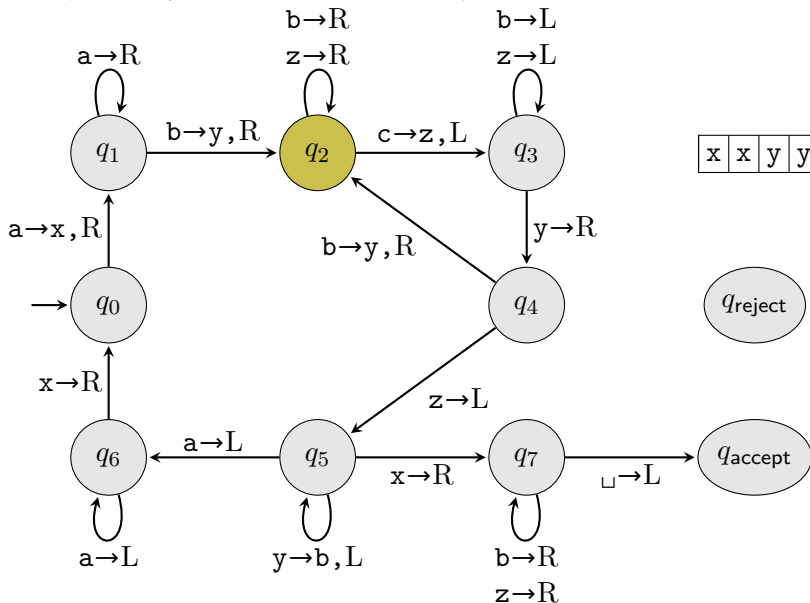
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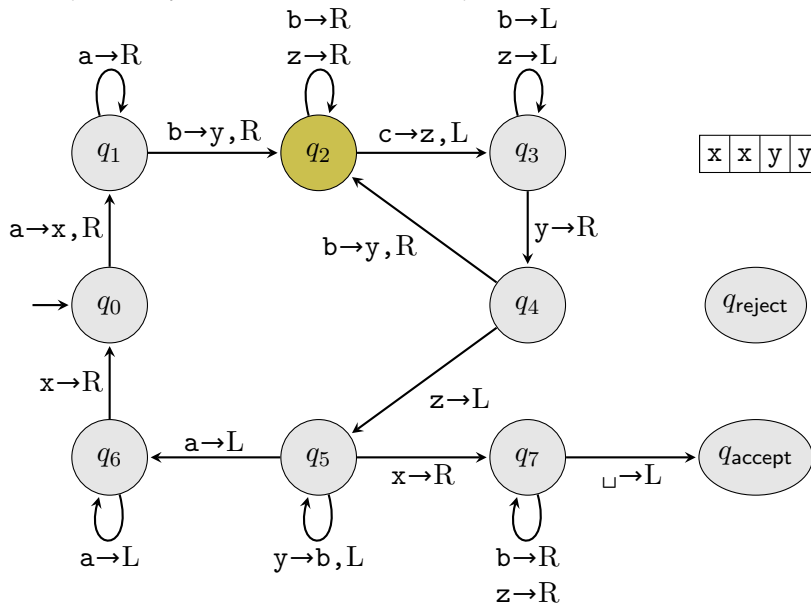
$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = \text{aabbbcccccc}$





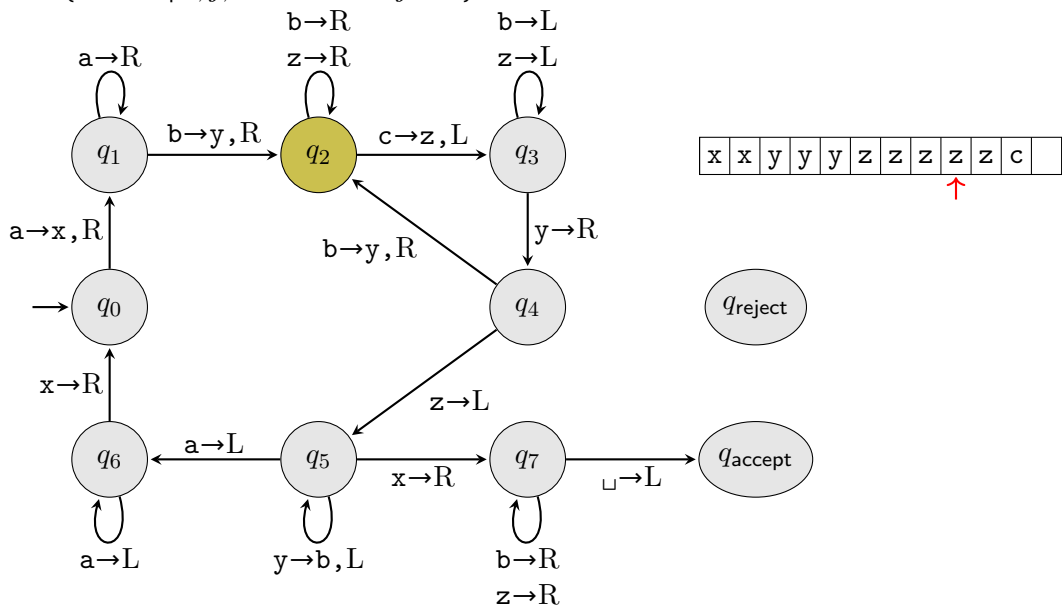
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$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = \text{aabbbcccccc}$



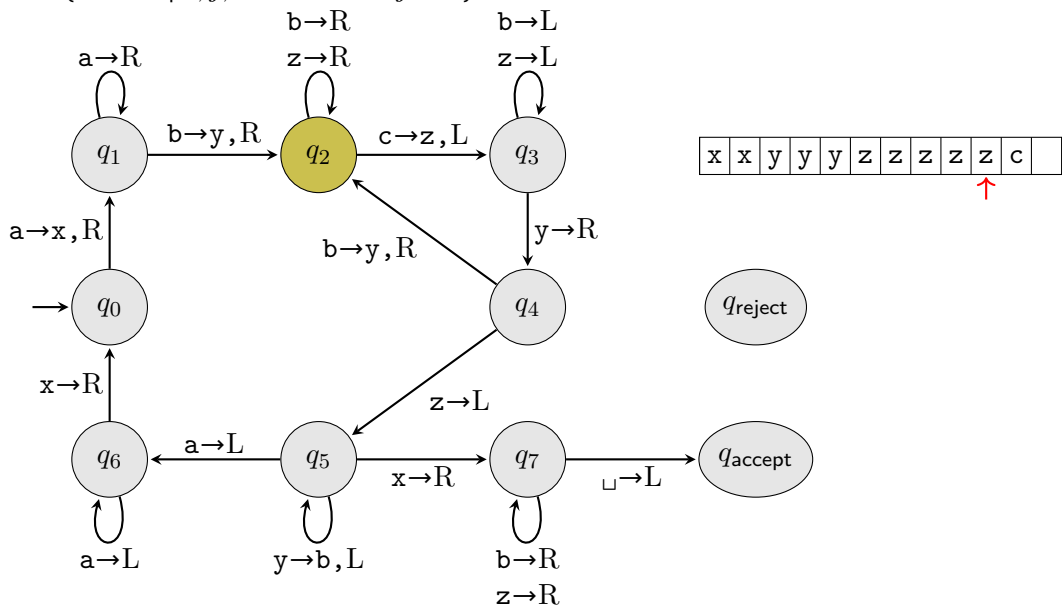
# Multiplication TM

$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = \text{aabbbcccccc}$



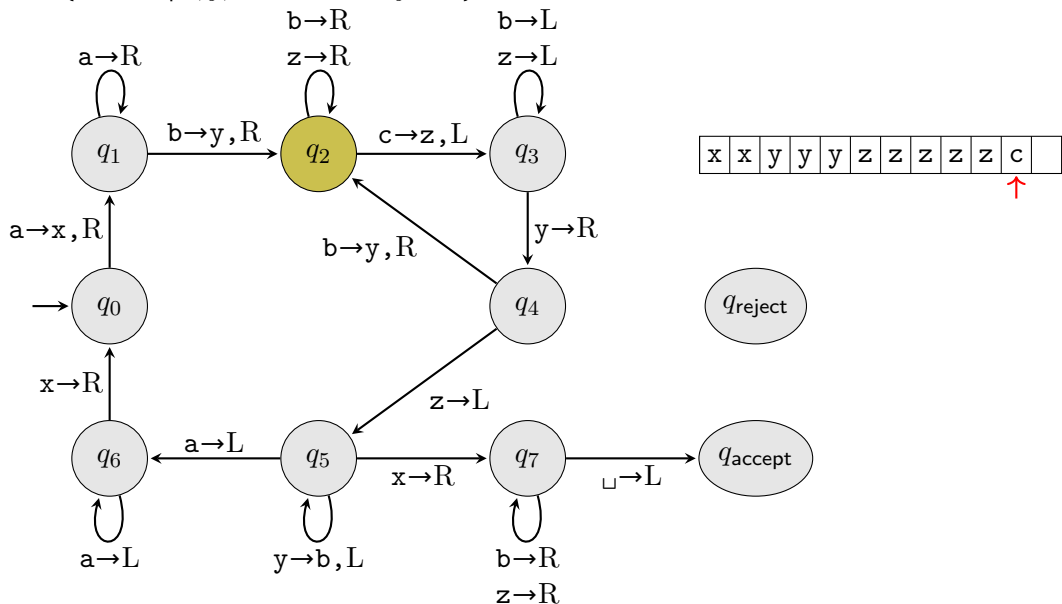
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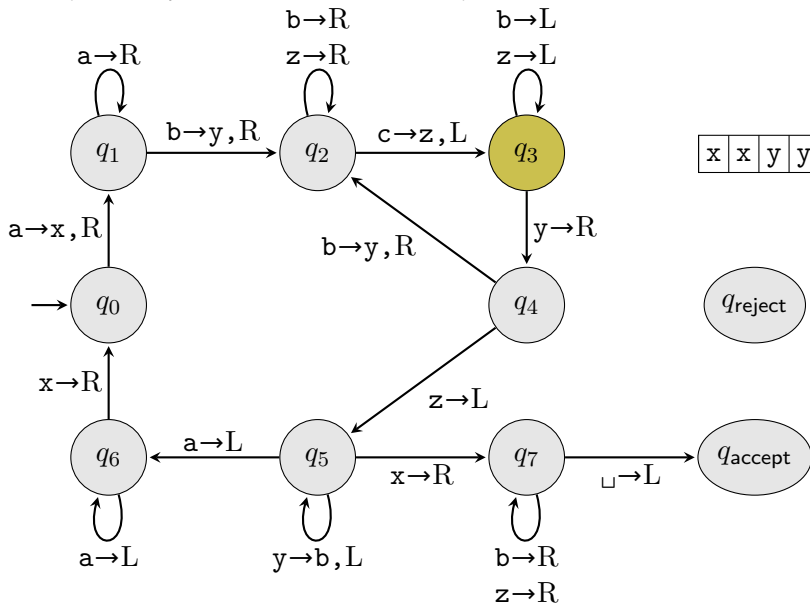
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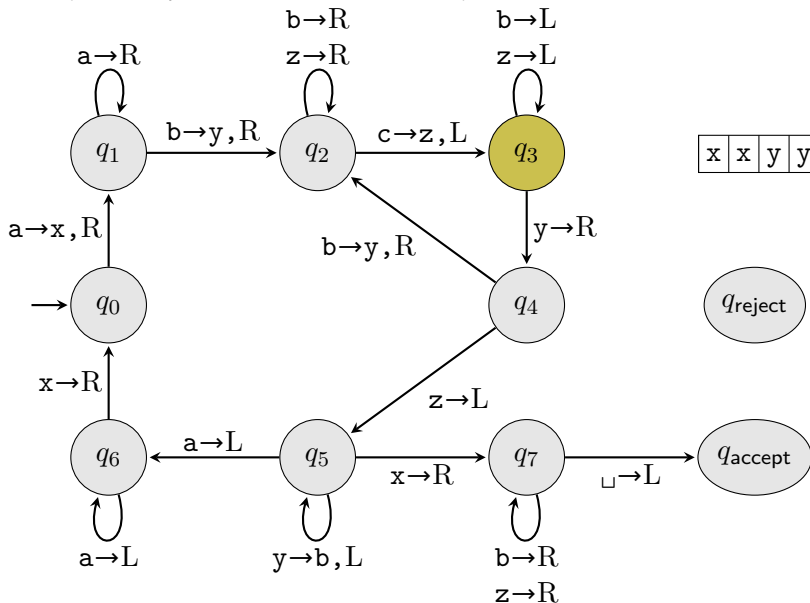
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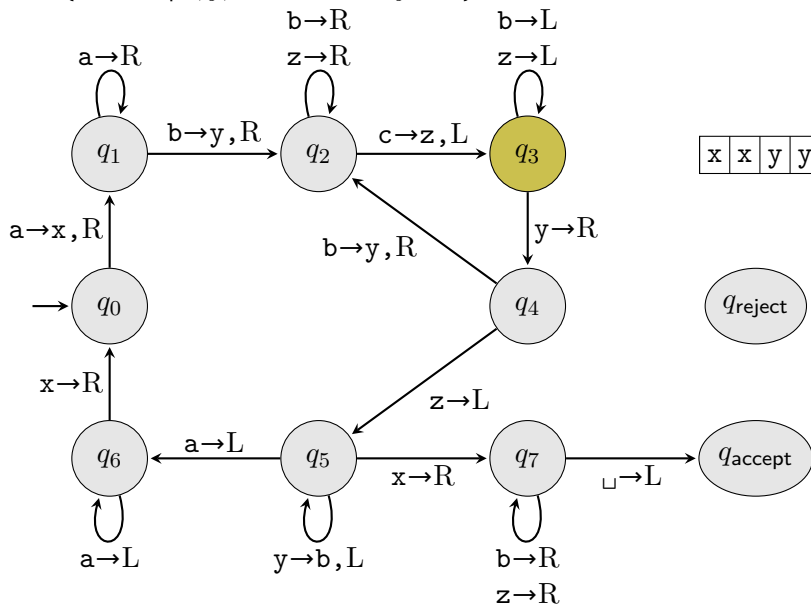
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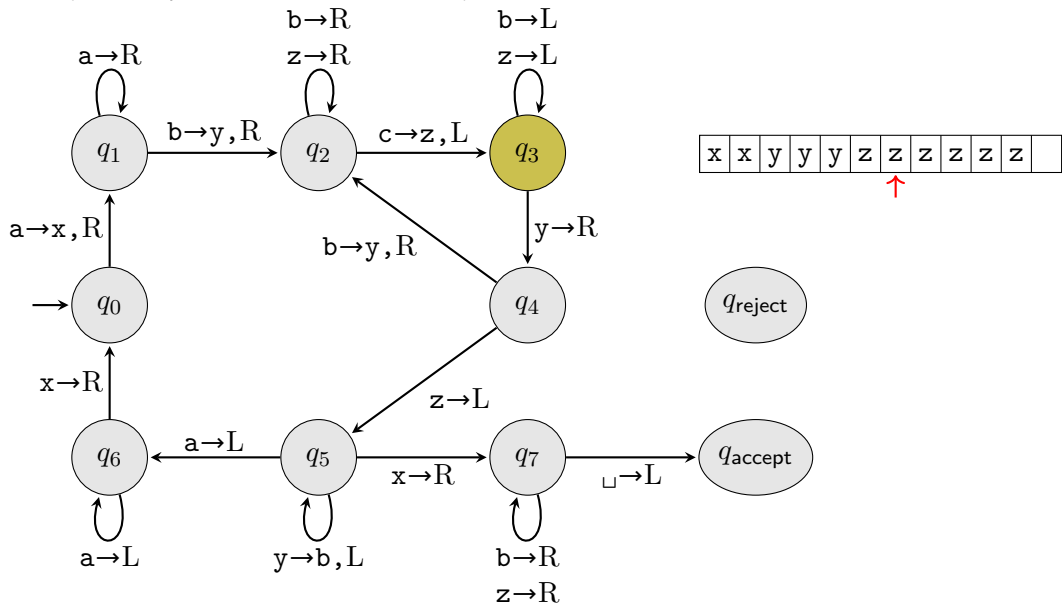
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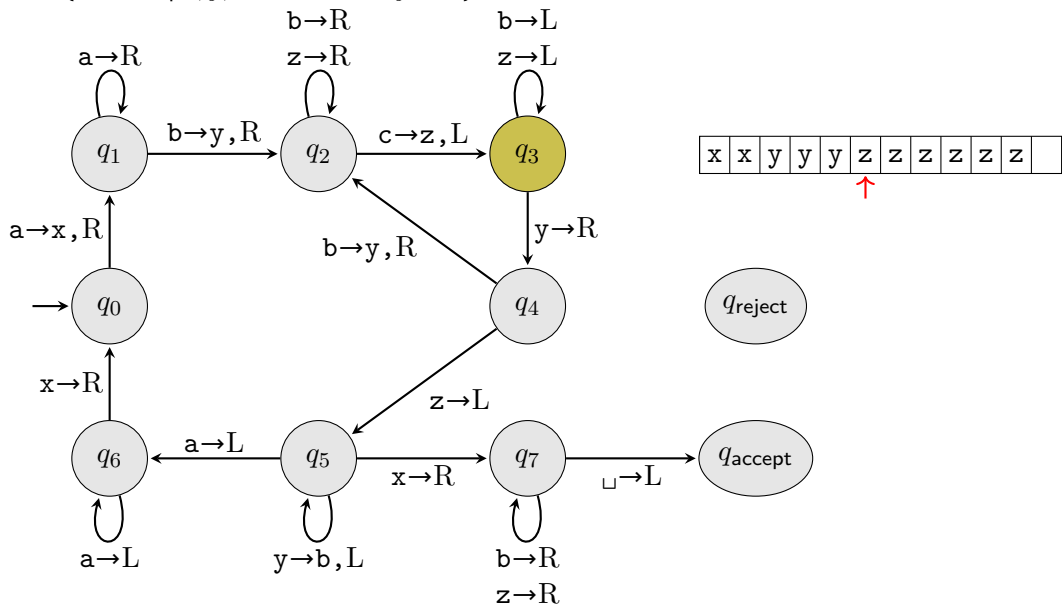
$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = \text{aabbbcccccc}$





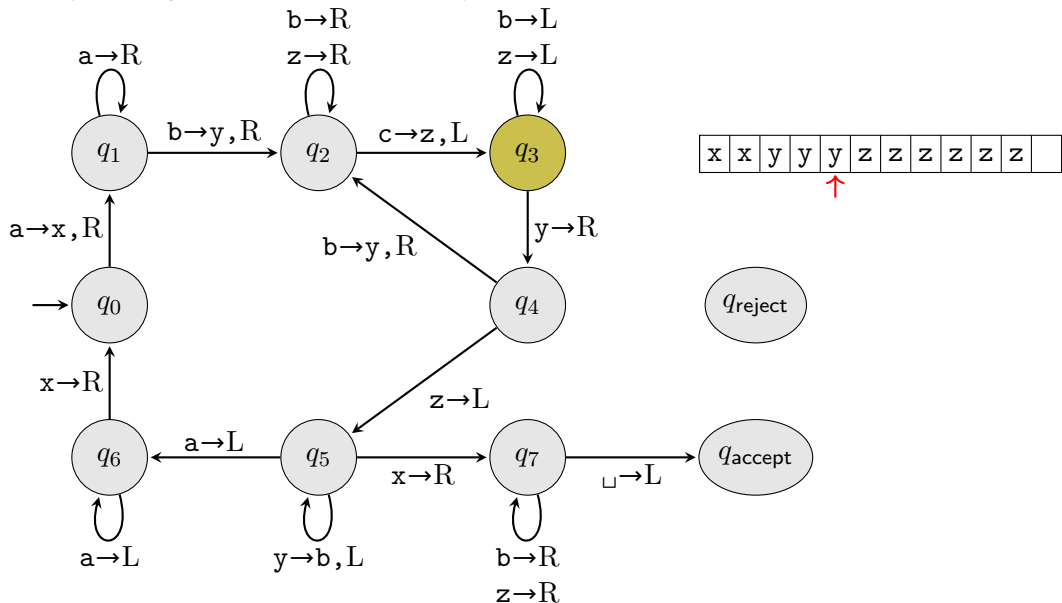
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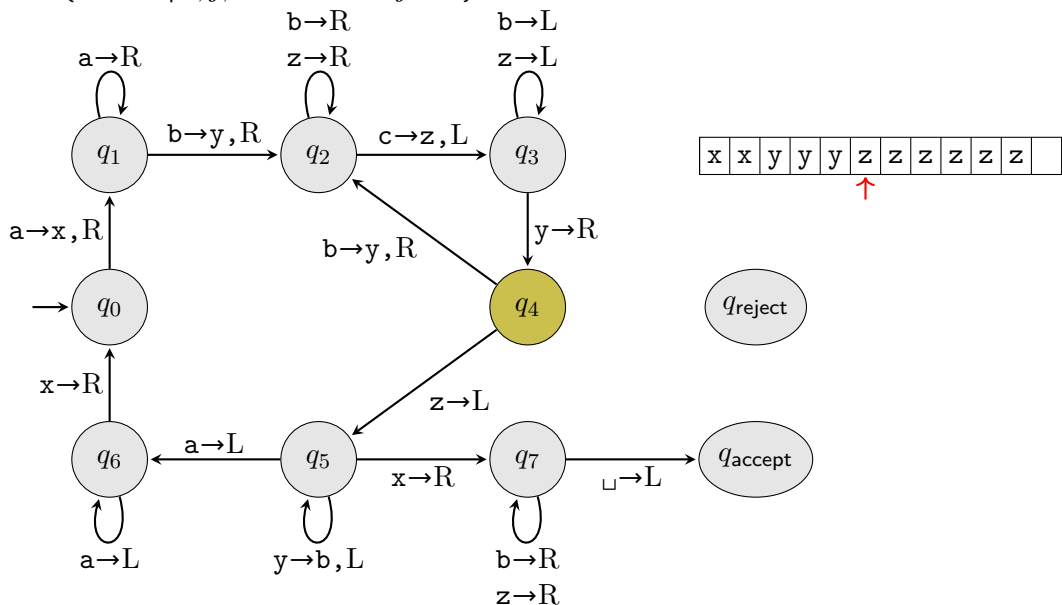
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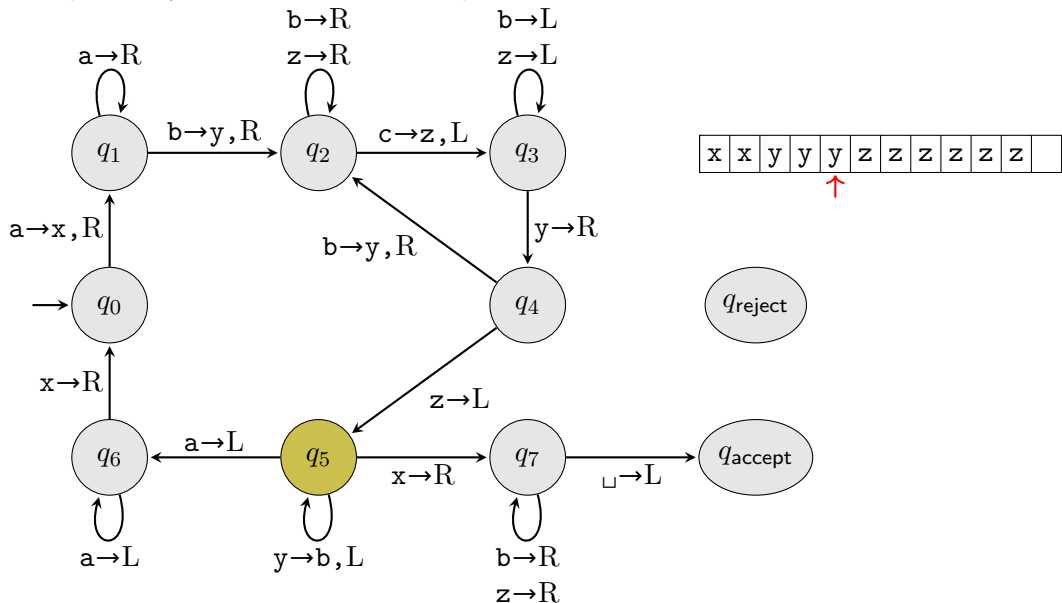
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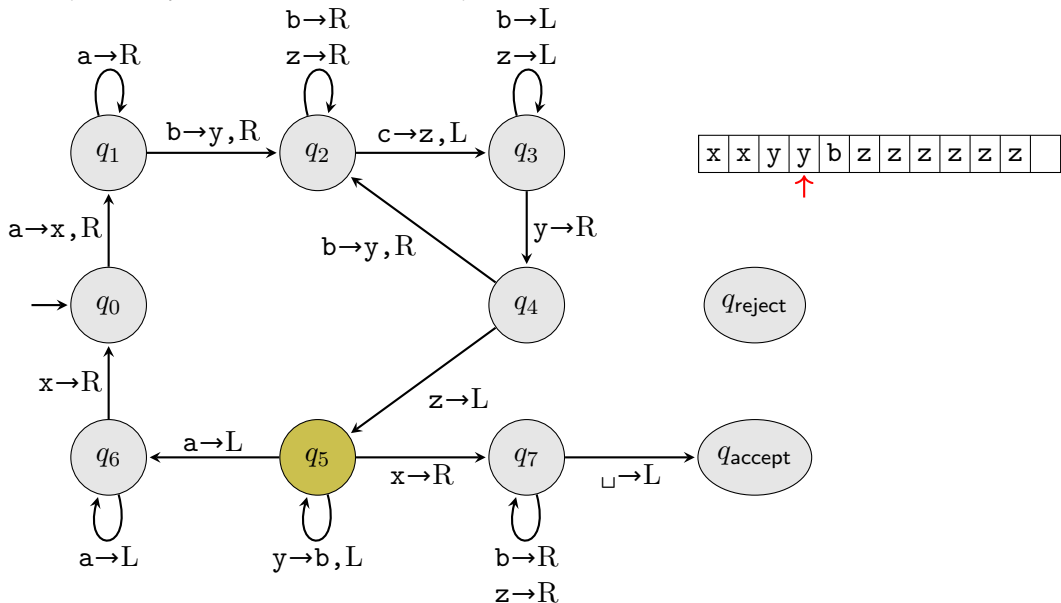
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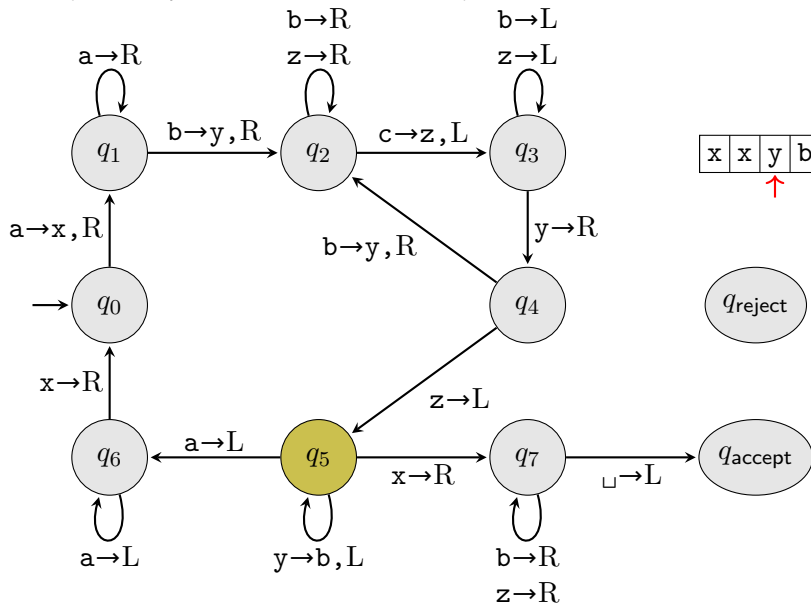
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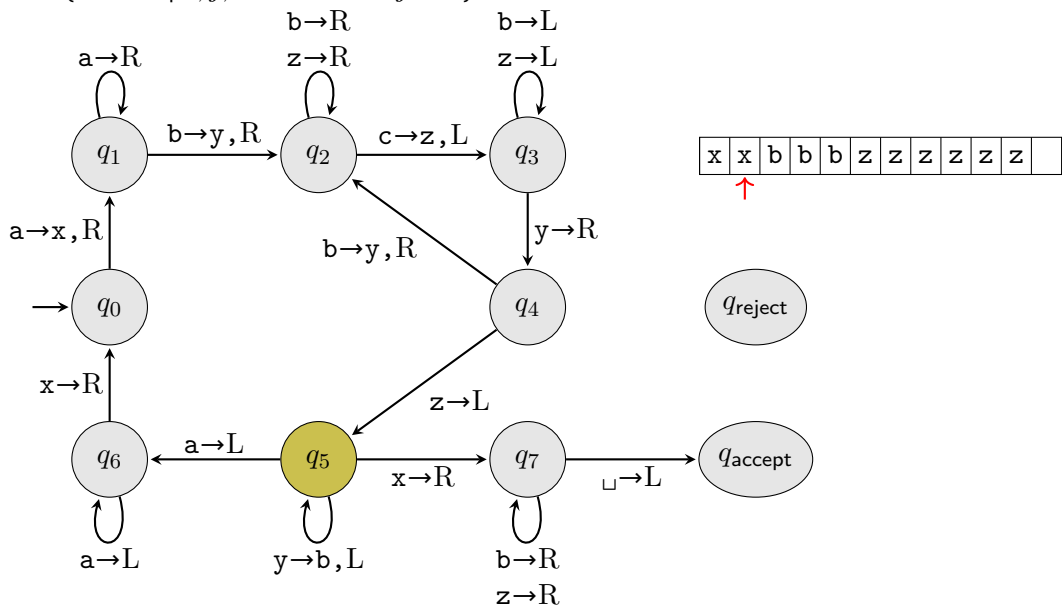
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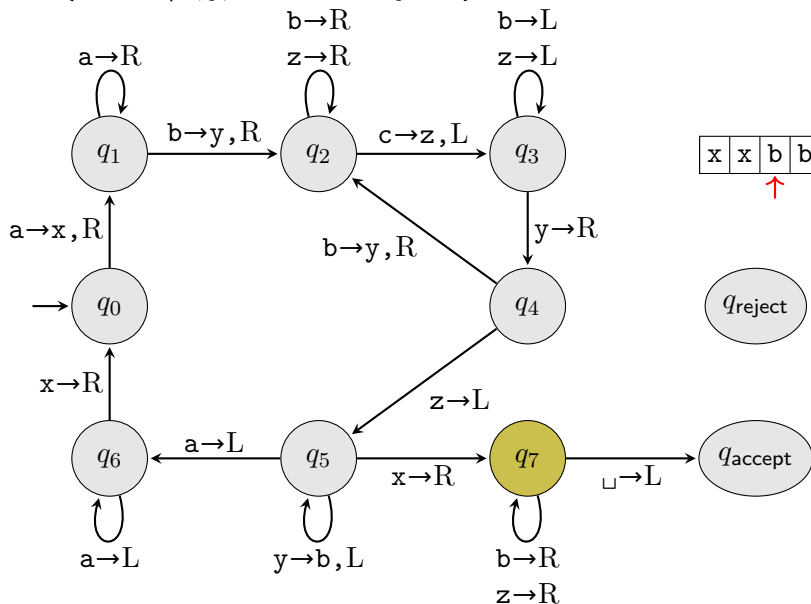
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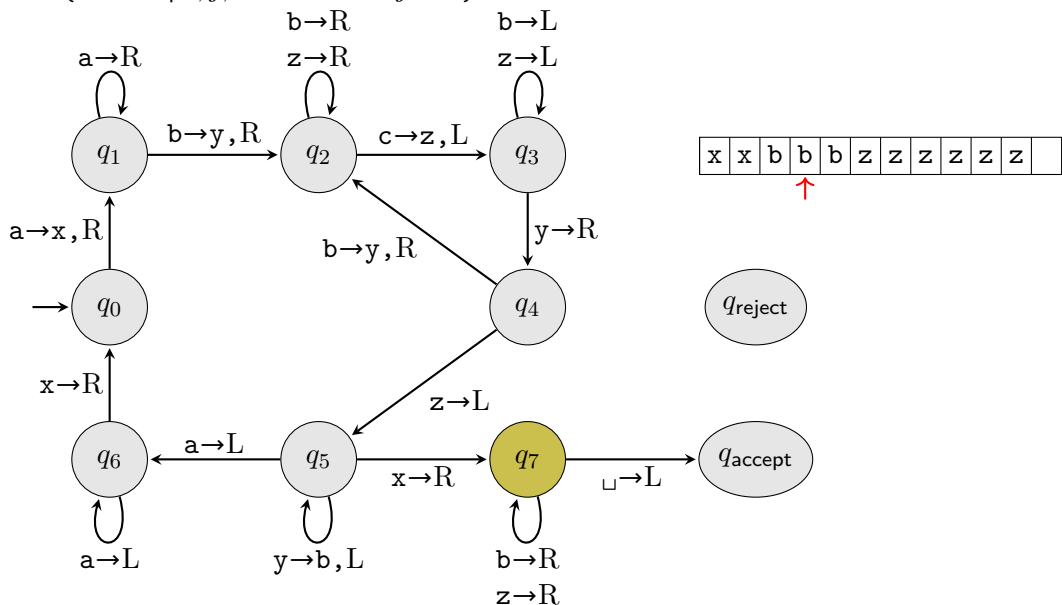
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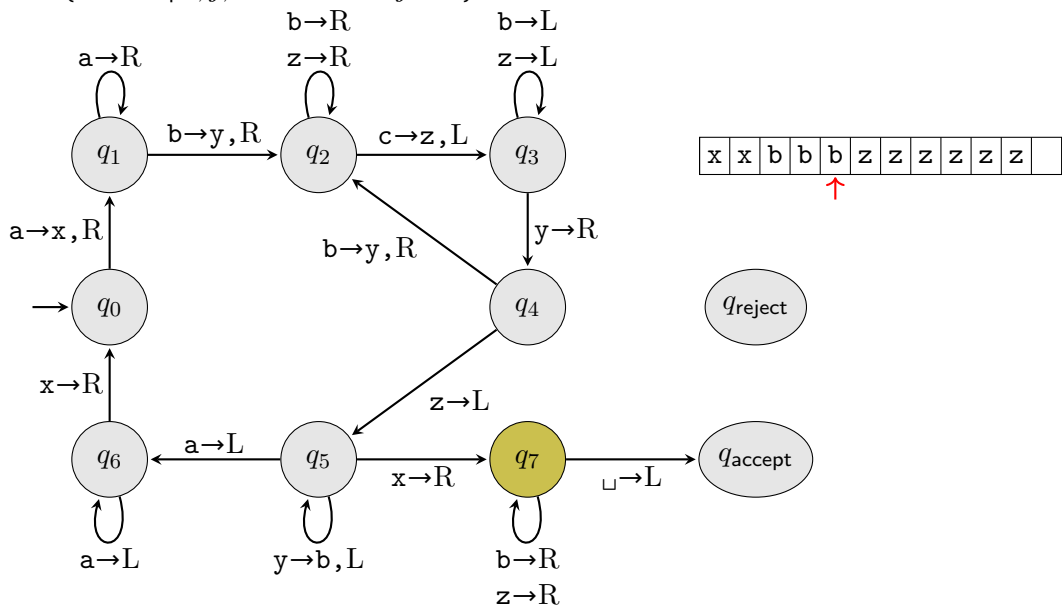
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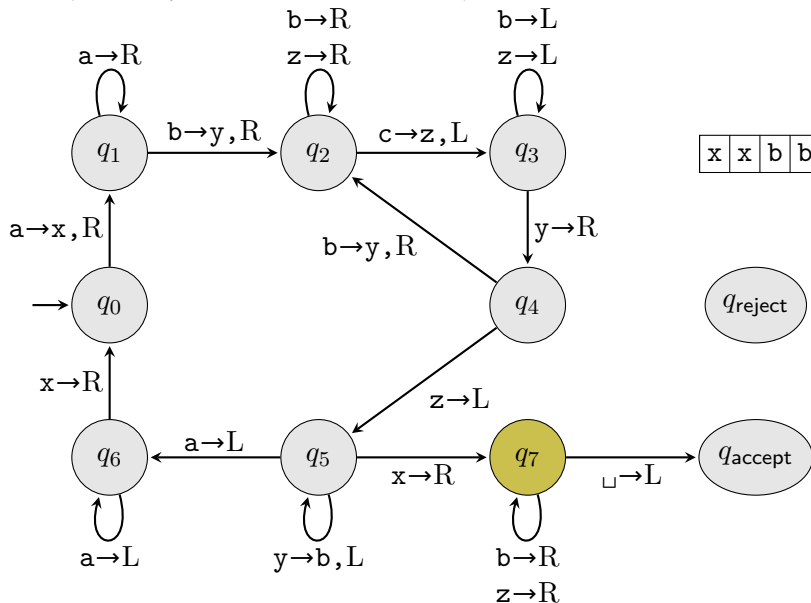
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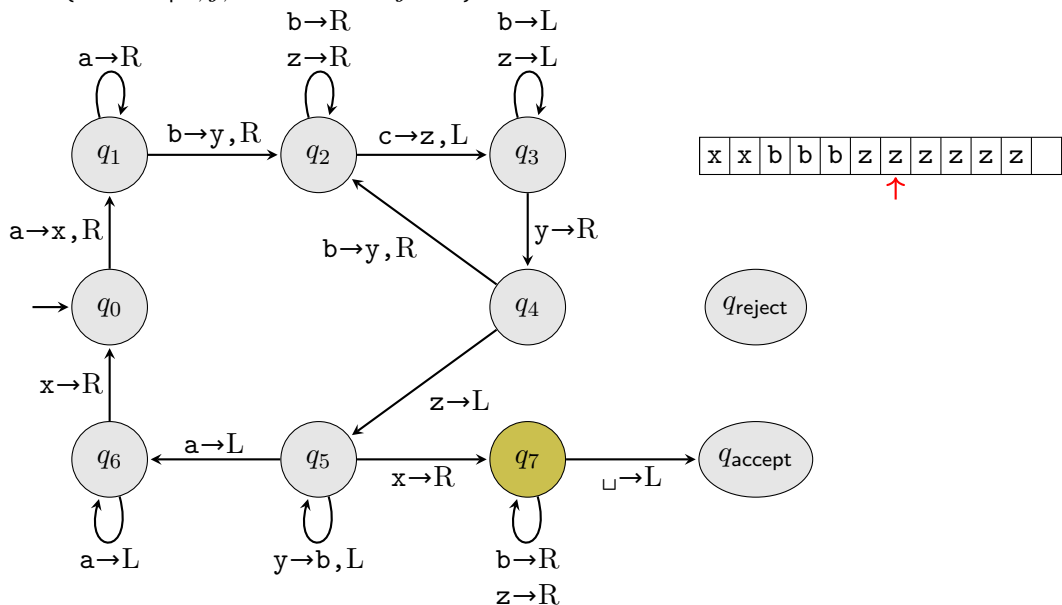
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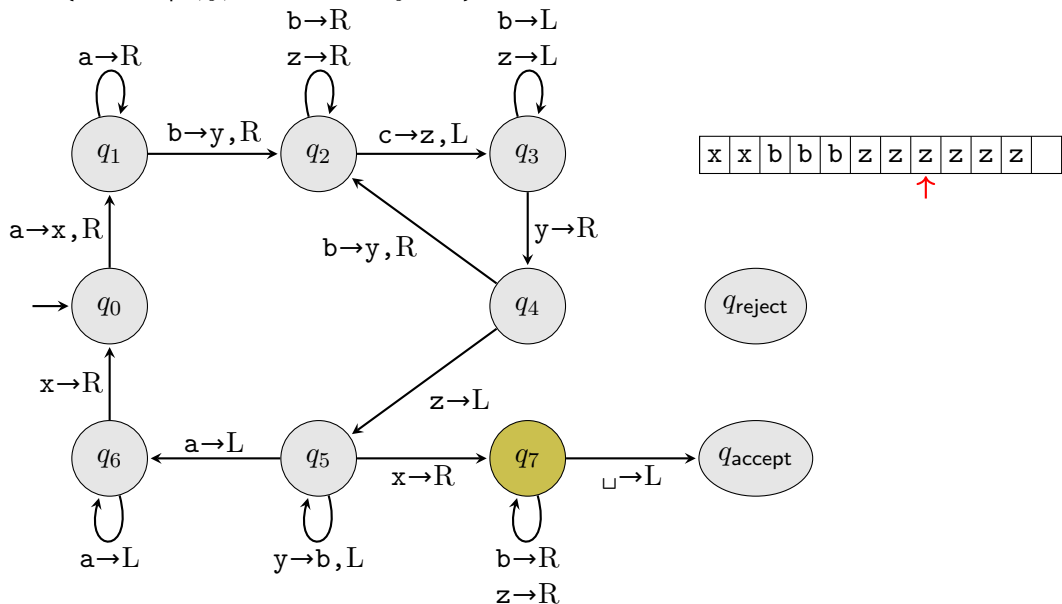
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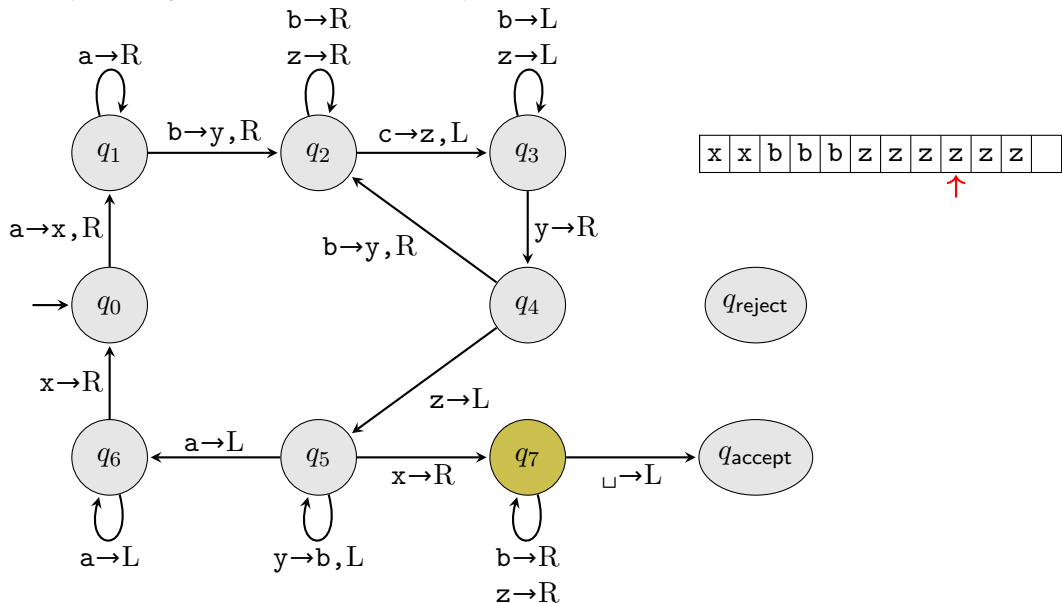
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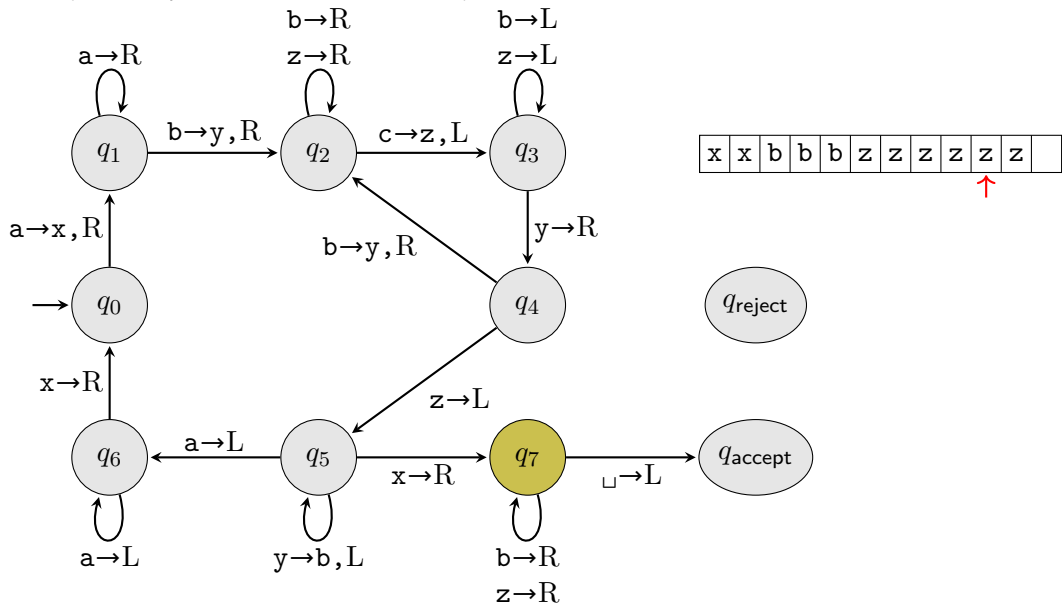
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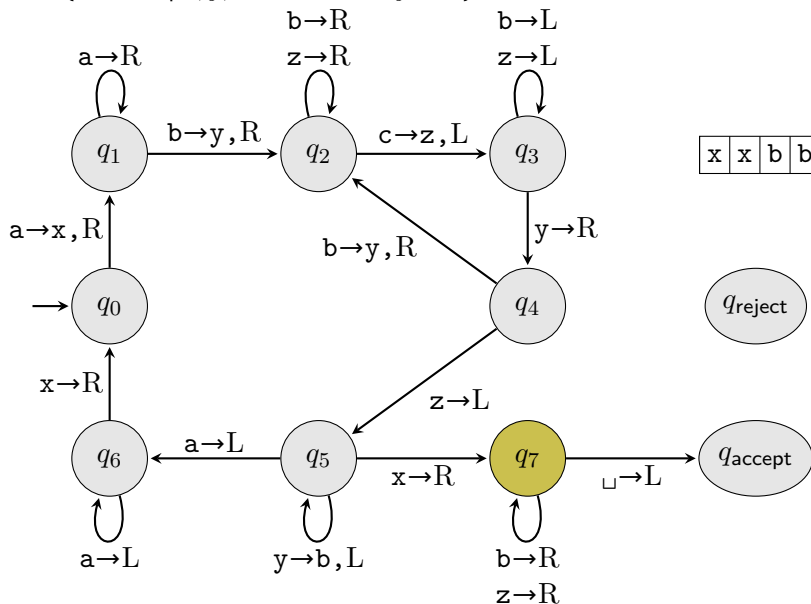






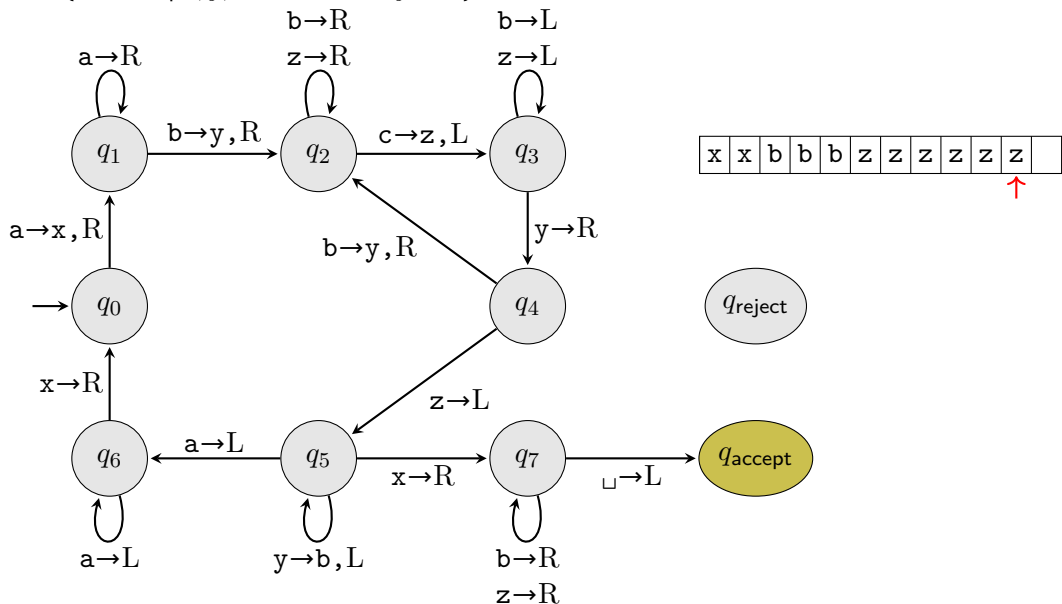
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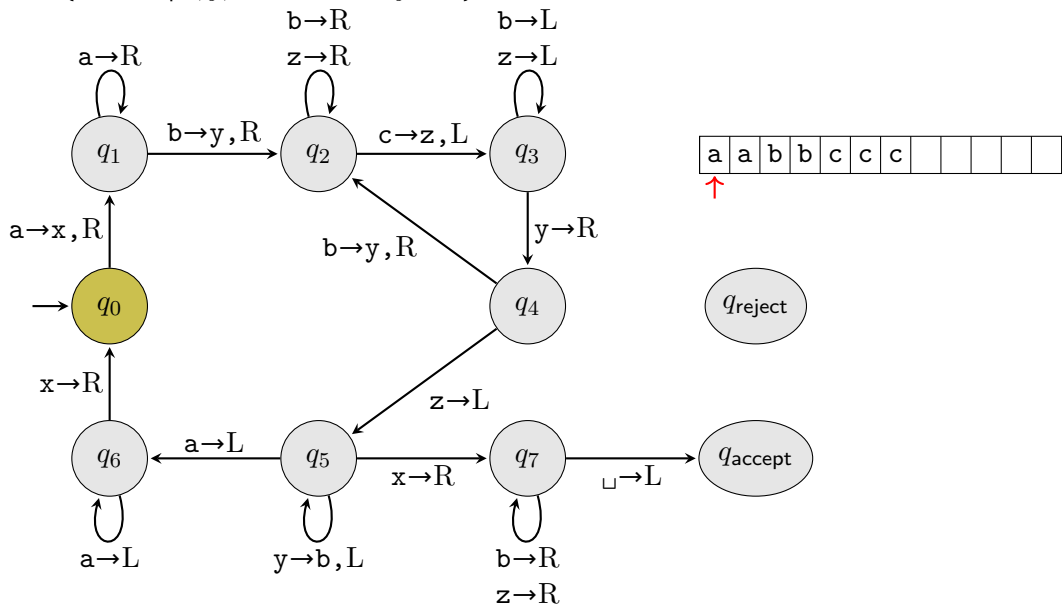
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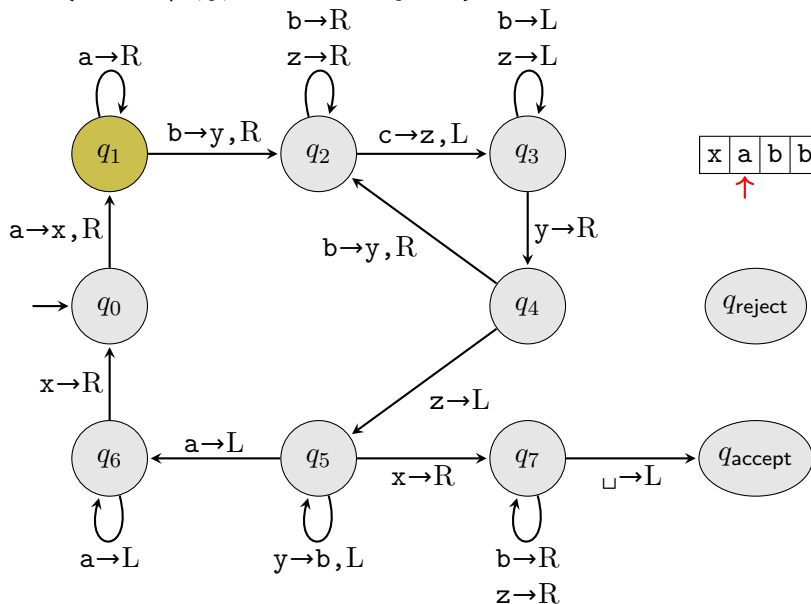
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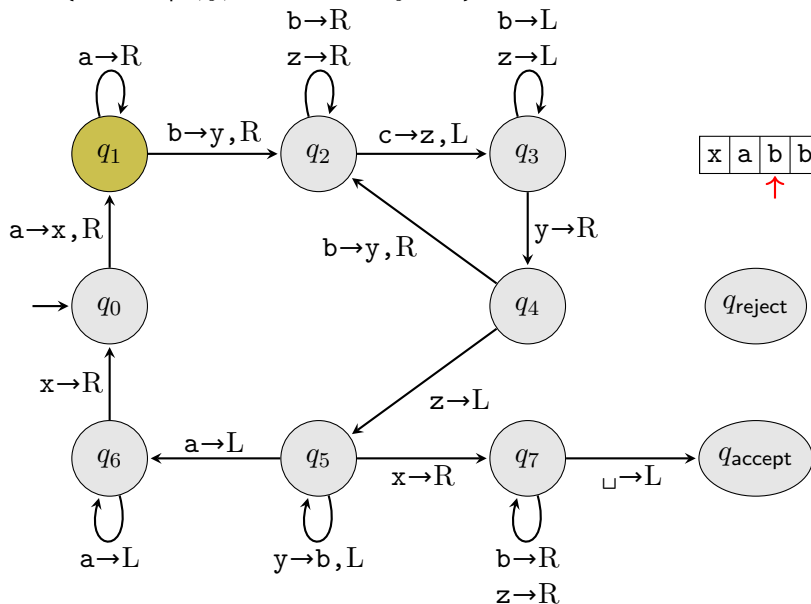
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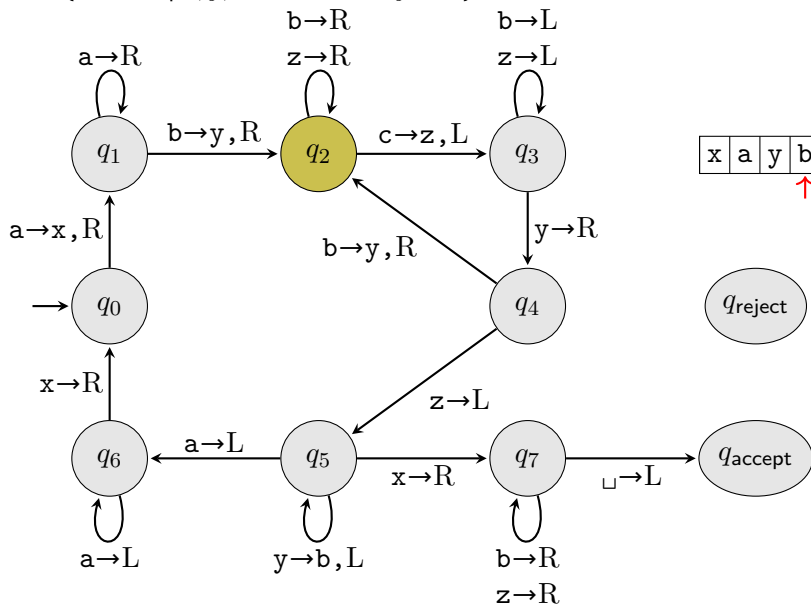
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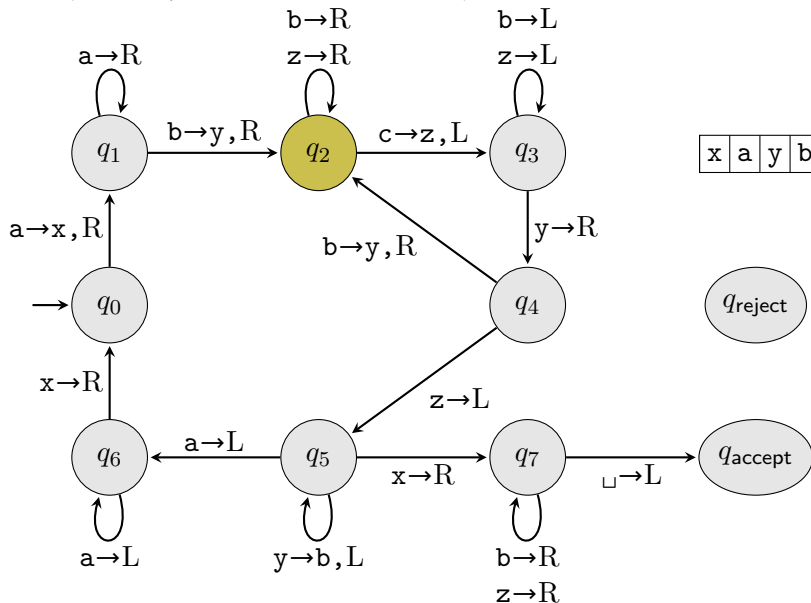
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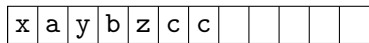
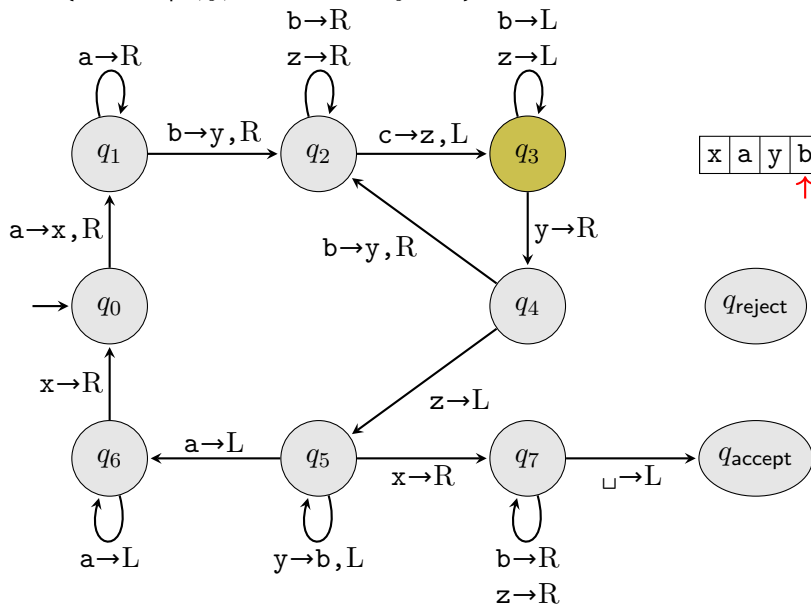
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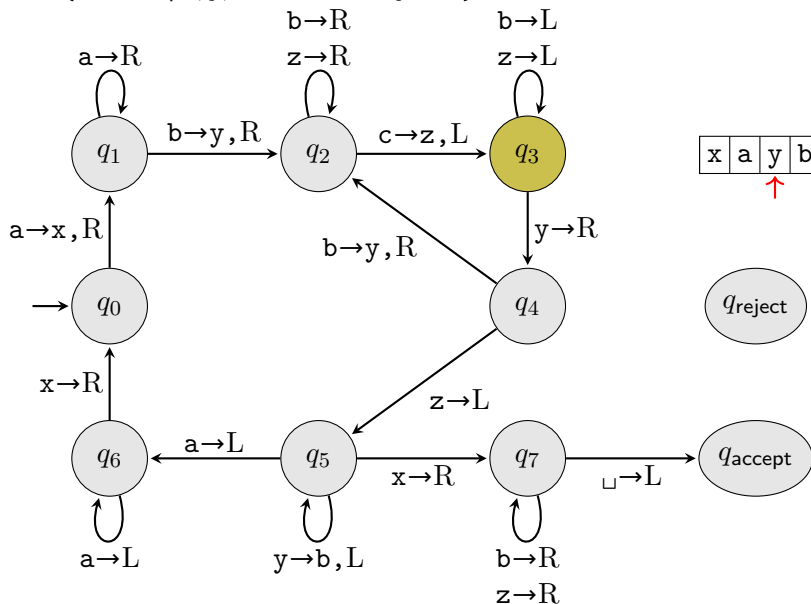
$B = \{a^i b^j c^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$      $w = aabbccc$





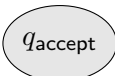
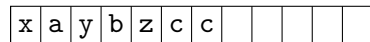
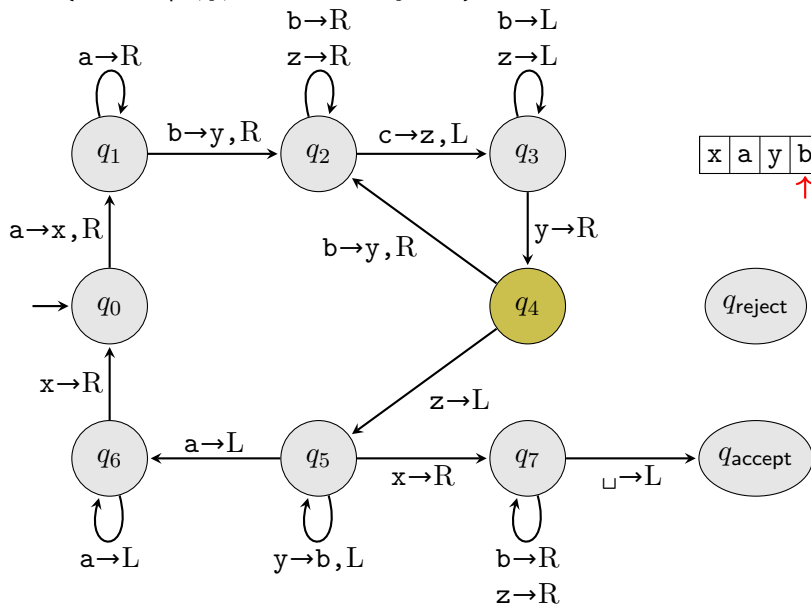
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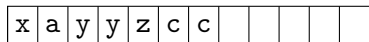
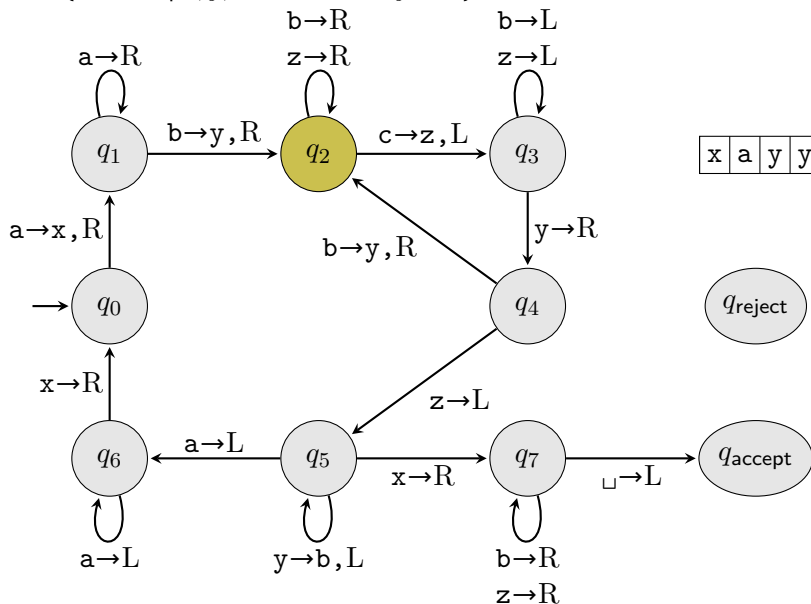
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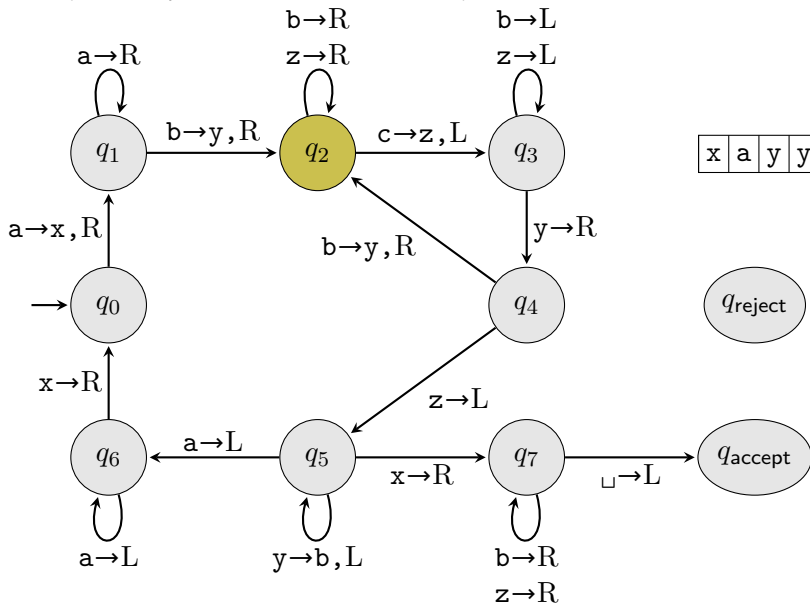
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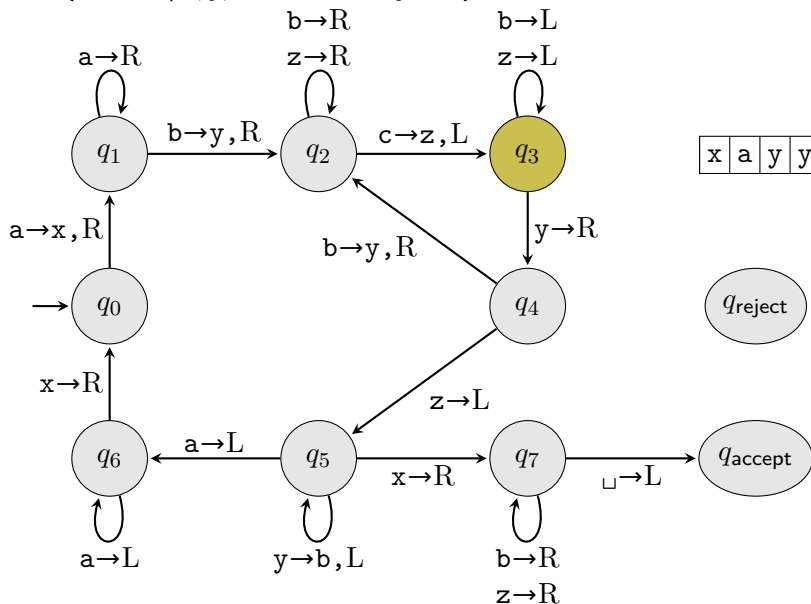
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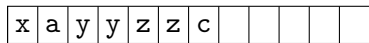
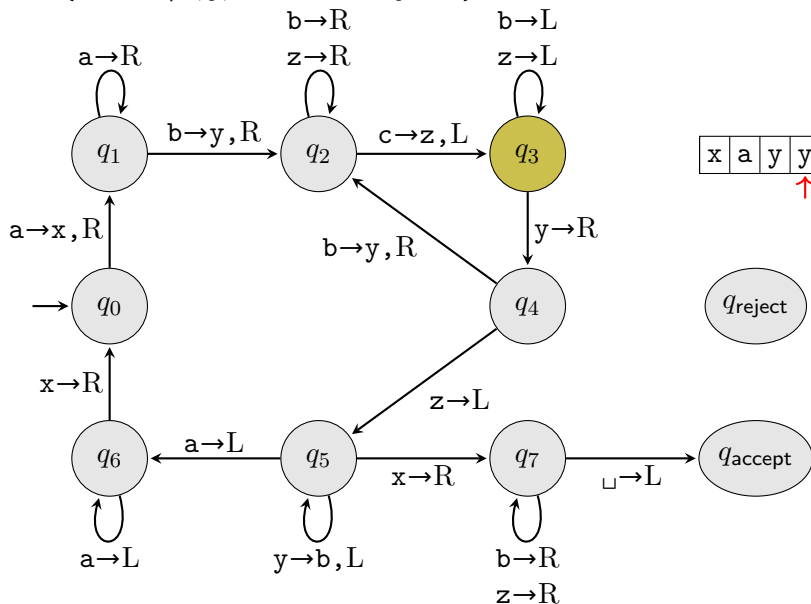
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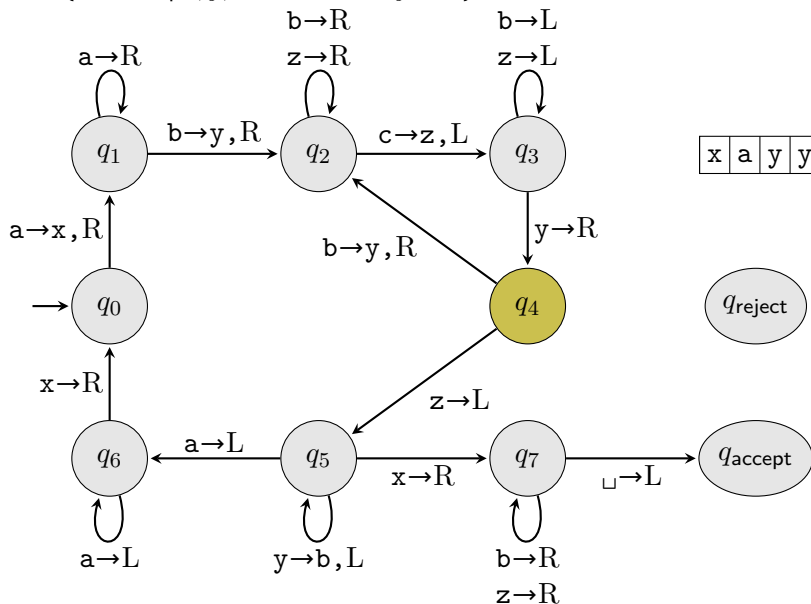
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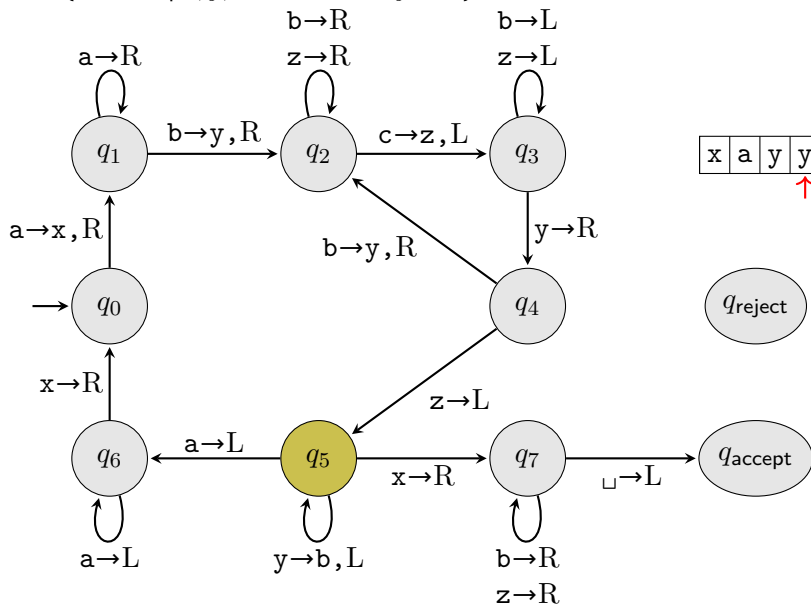
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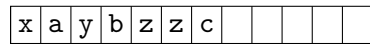
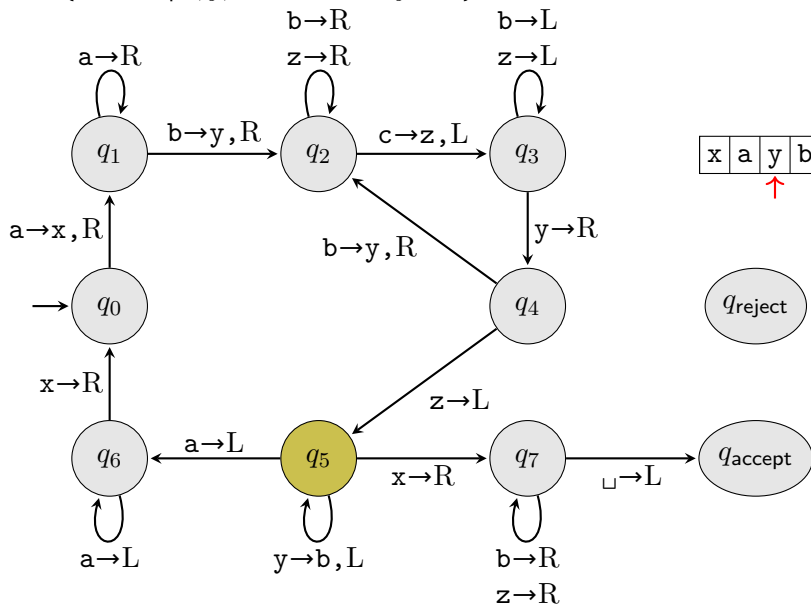
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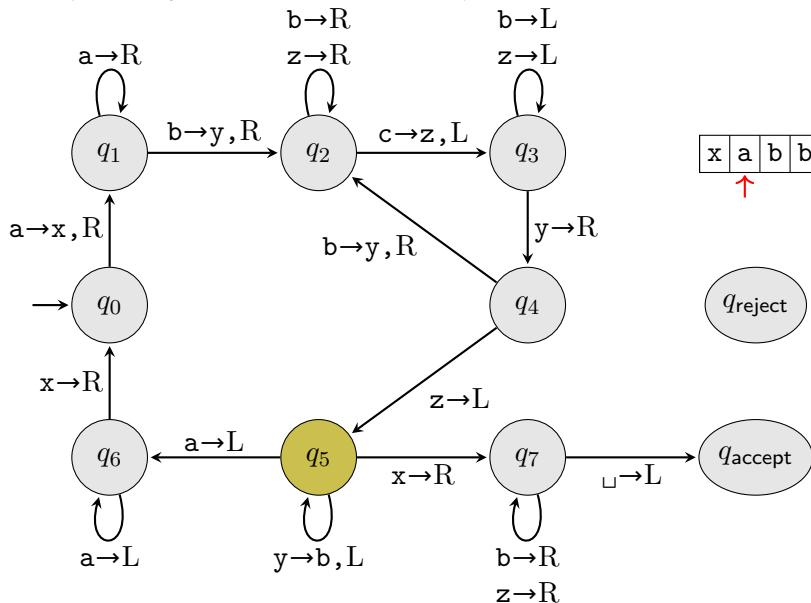
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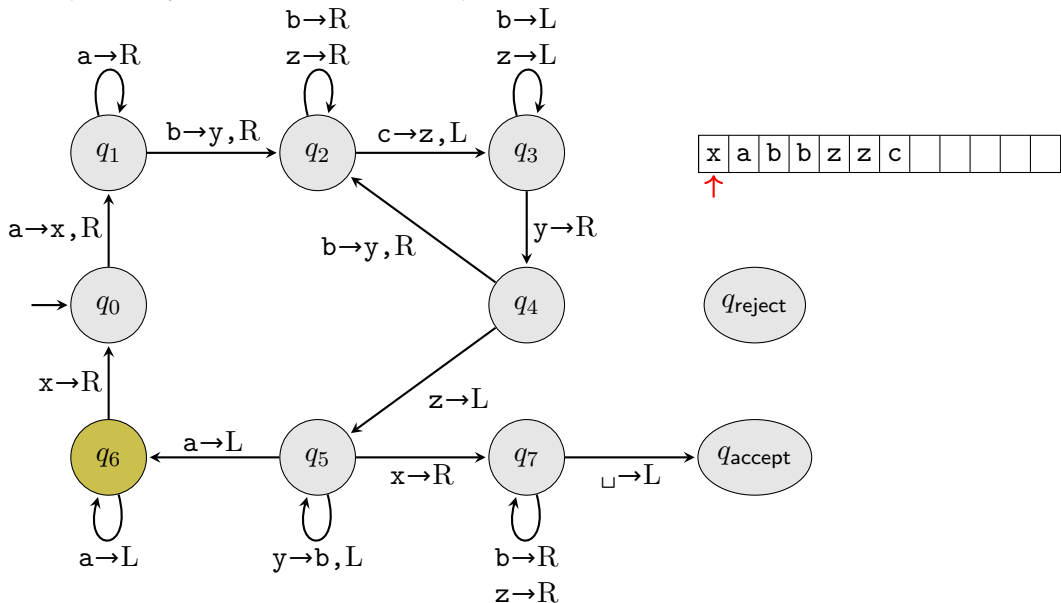
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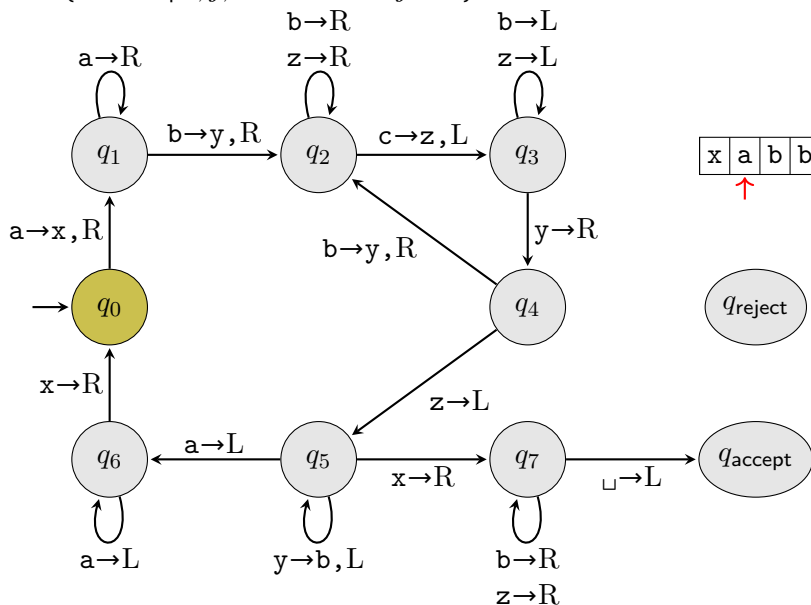
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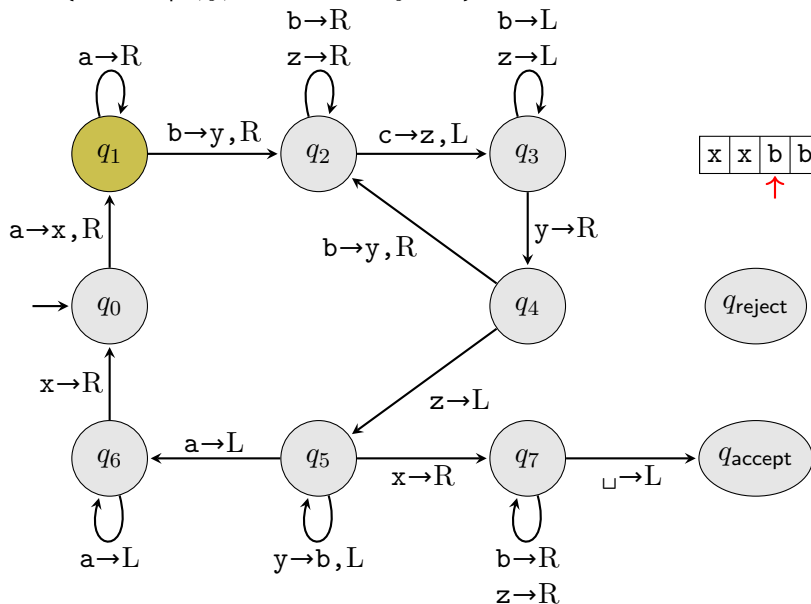
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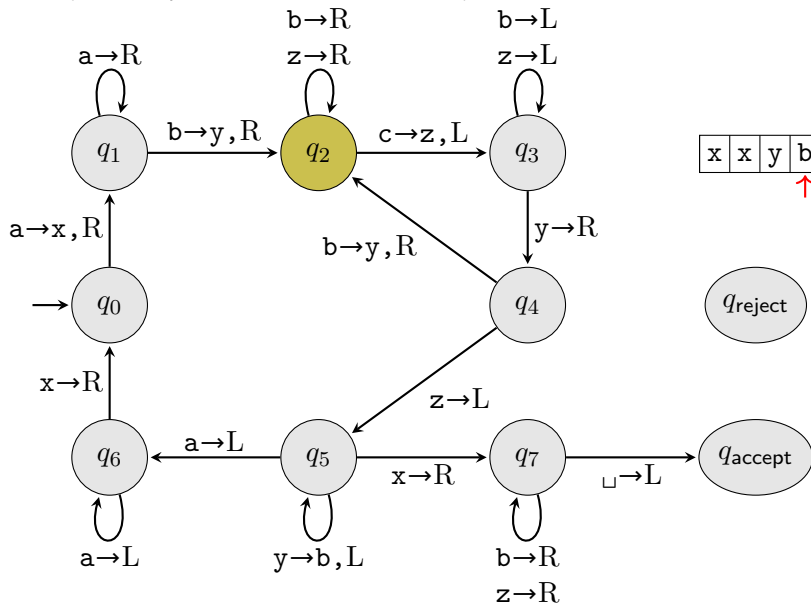
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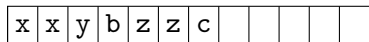
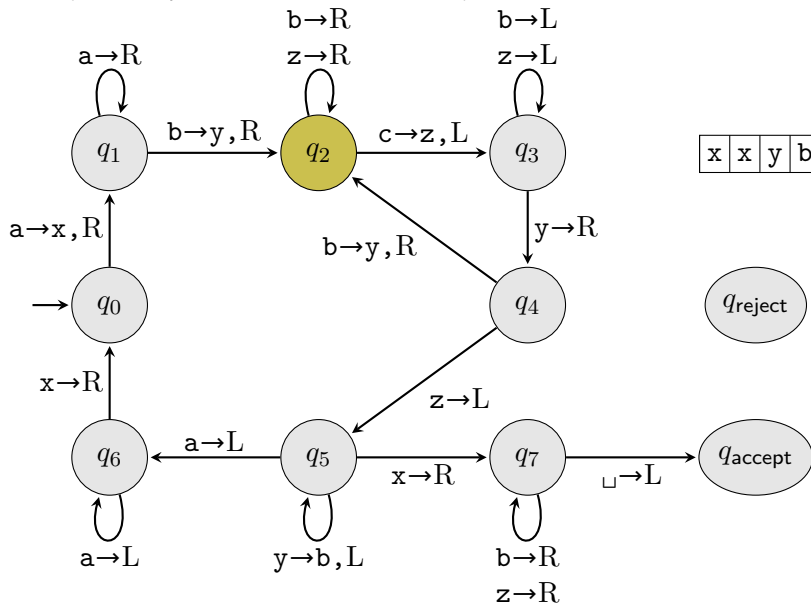
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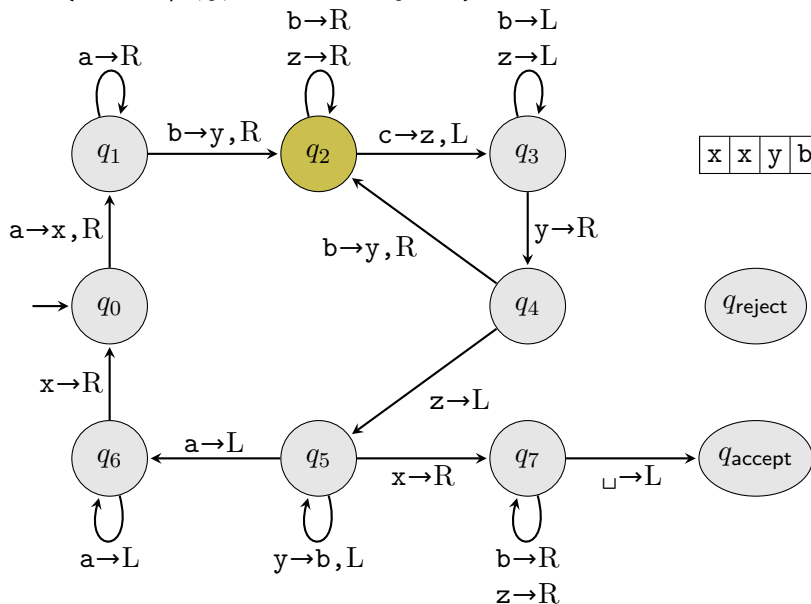
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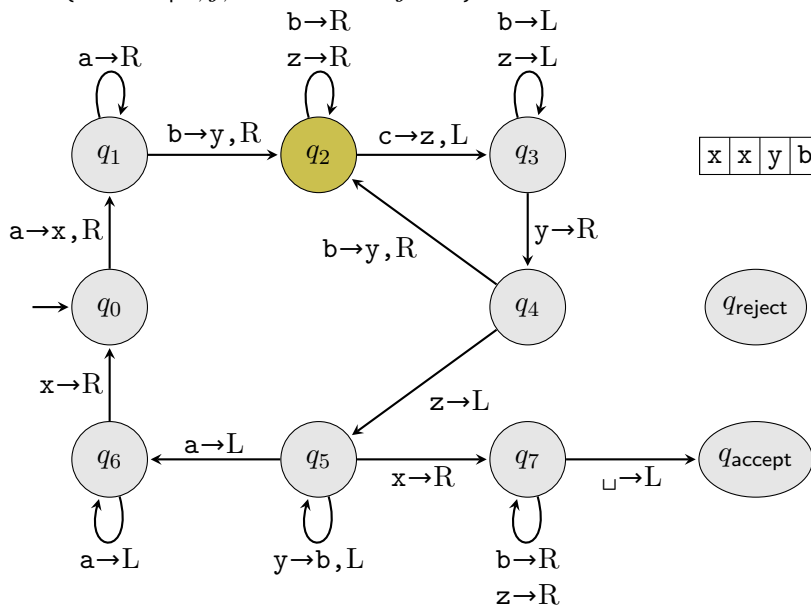
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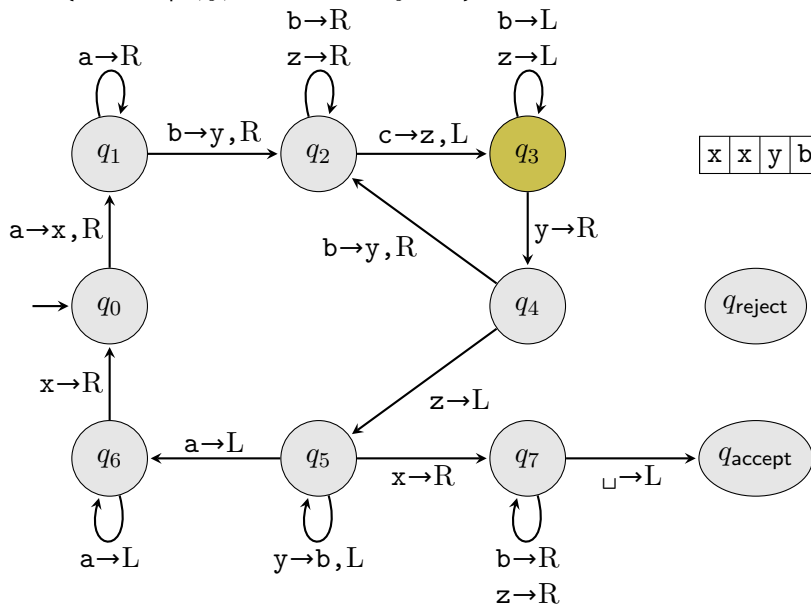
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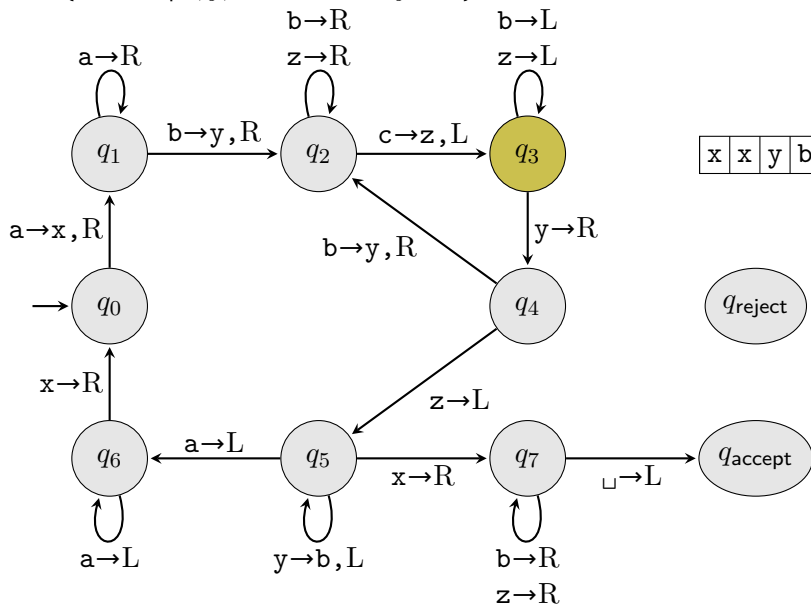
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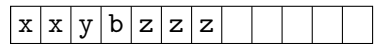
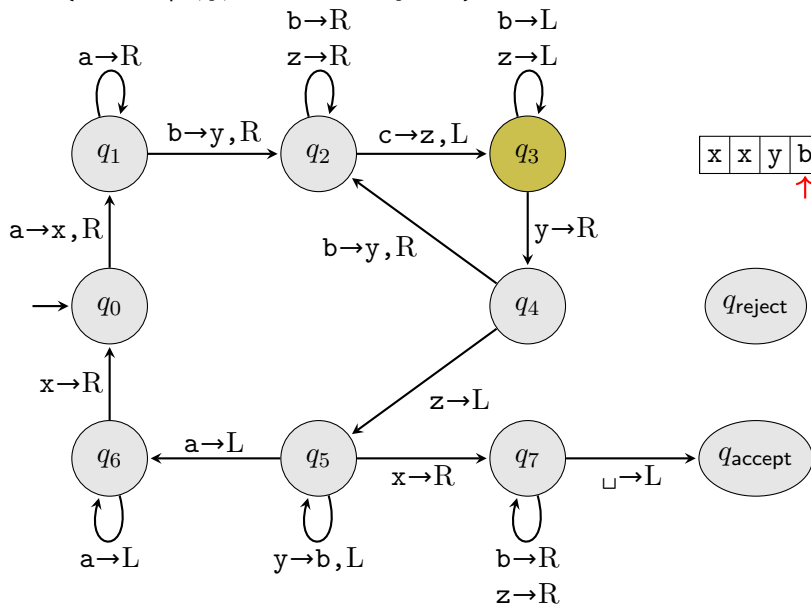
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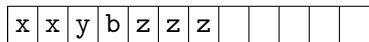
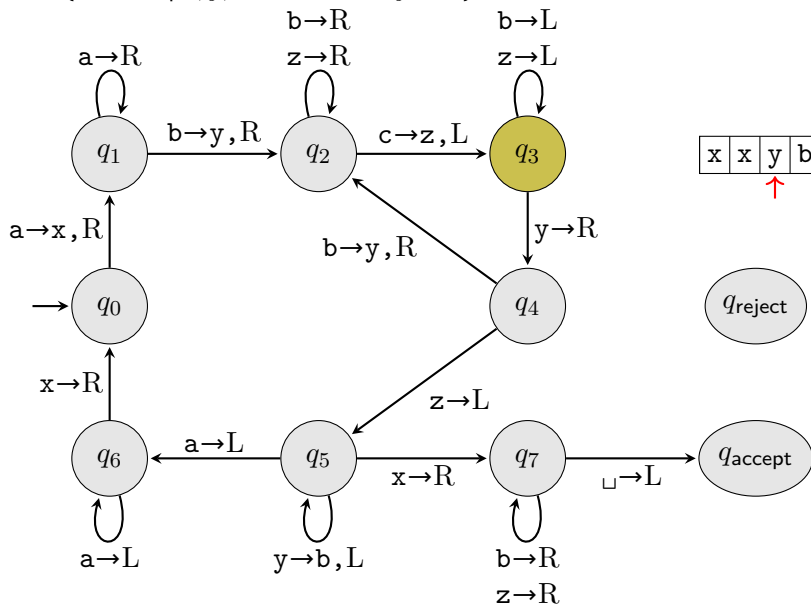
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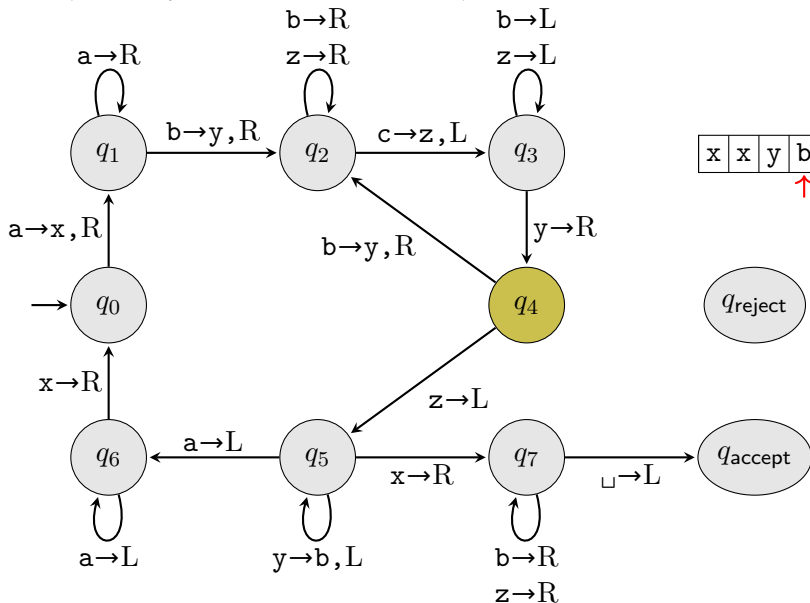
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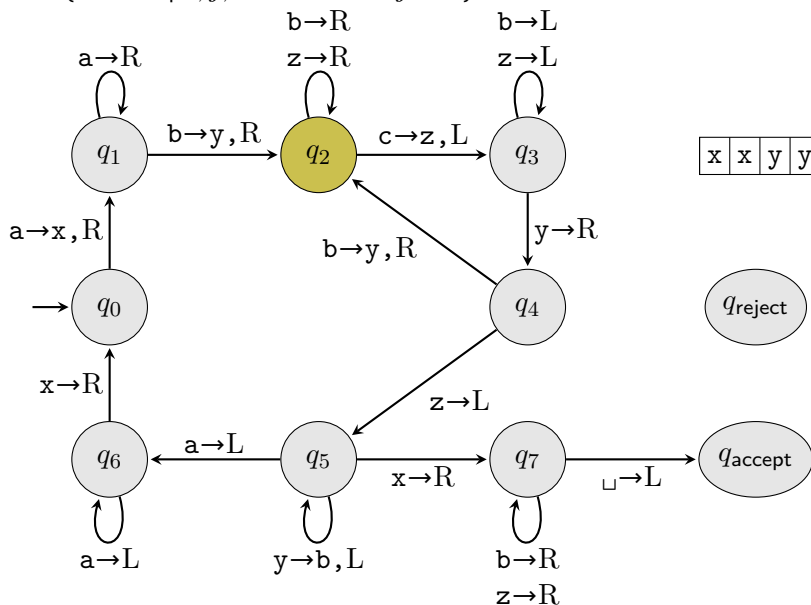
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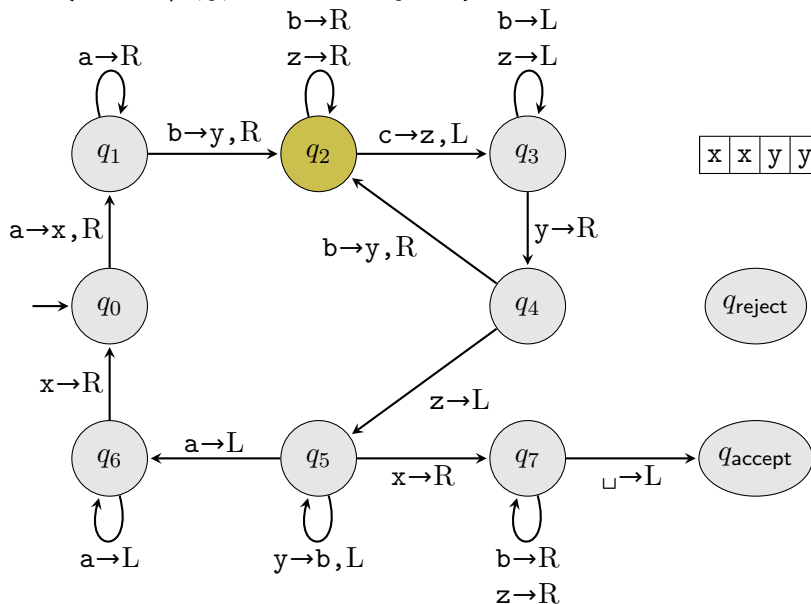
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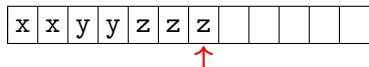
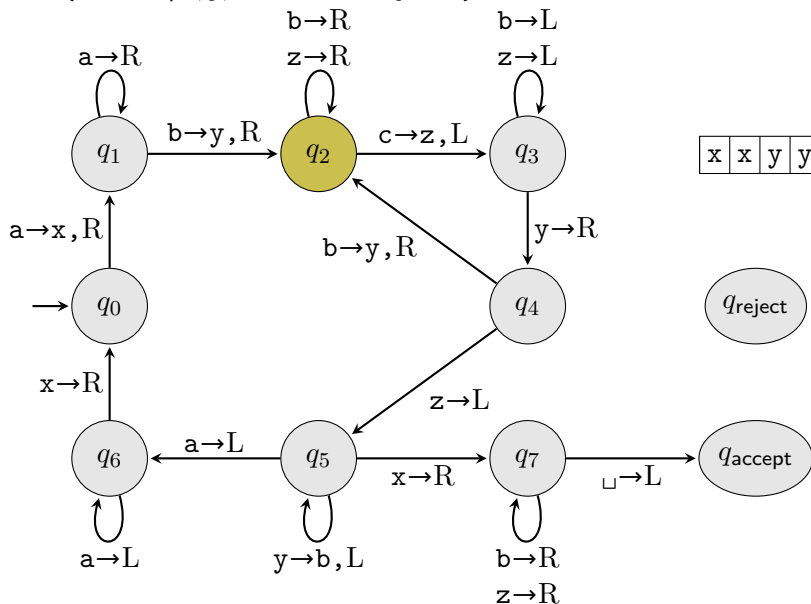
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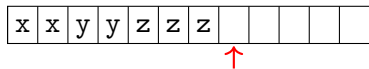
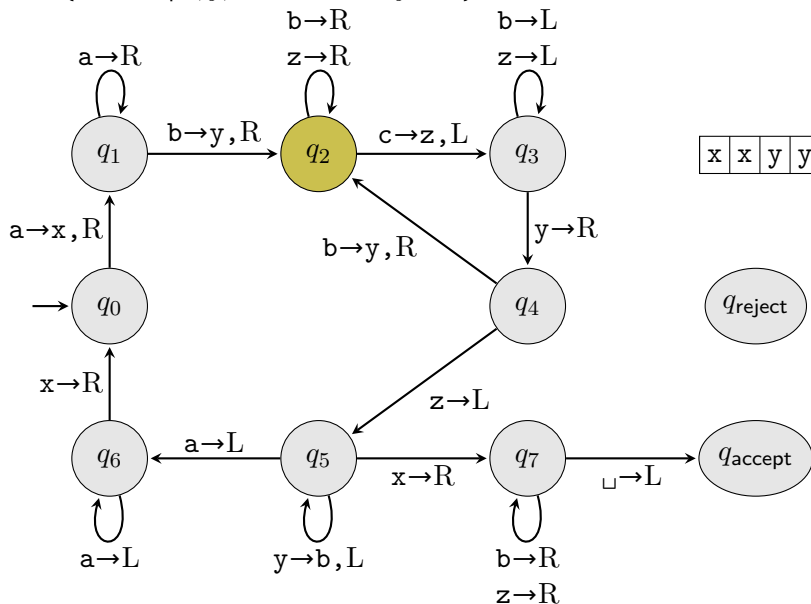
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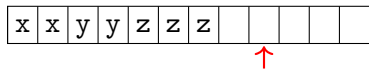
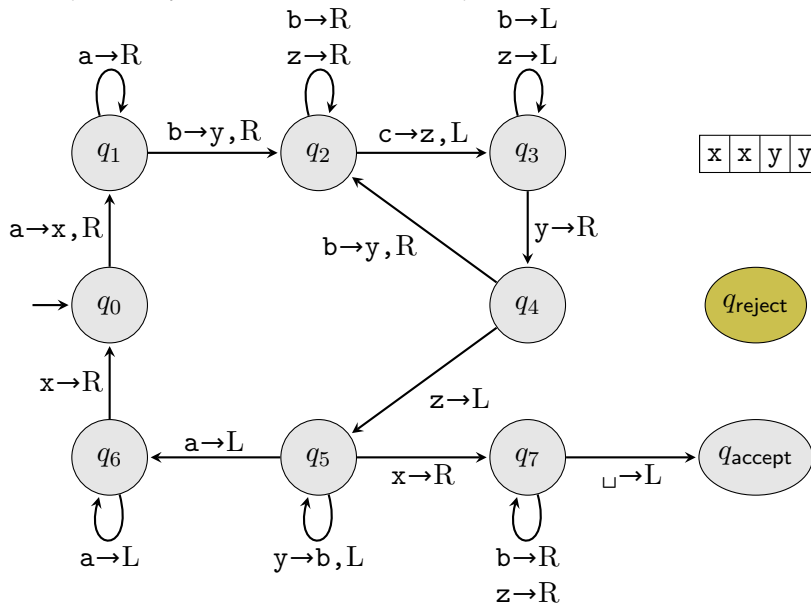
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## Higher-level descriptions of TMs

Only the simplest TMs are reasonable to actually construct diagrams for

Instead, we want to give a higher-level description of a TM

For now, we'll stick with “implementation-level” descriptions, but soon we'll describe high-level algorithms

# Implementation-level descriptions of TMs

Give a sequence of steps that a TM would follow

Examples

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Give a sequence of steps that a TM would follow

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- “Check that the input matches some regular expression”

This is easy because we could convert the regex to a DFA and encode the DFA in the TM. The TM would make a single pass over the input (without changing anything) and check if the DFA would accept

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For each symbol  $t \in \Gamma$  we need to mark, add another symbol, say  $\dot{t}$  to  $\Gamma$

- “Return to the beginning of the tape”

More complicated but we have several options:

- ① Mark the first cell. E.g., for each  $t \in \Sigma$ , include both  $t$  and  $\dot{t}$  in  $\Gamma$ ; start the TM by replacing the first cell with its dotted version
- ② Copy the input right one cell and put a special marker like \$ in the first cell
- ③ Use the fact that when TMs try to move left in the left-most cell, they stay in place: Remember the current symbol (by using states), replace it with \$, move left, if the cell remains \$, it's at the left-most cell. In any case, replace the original cell's symbol

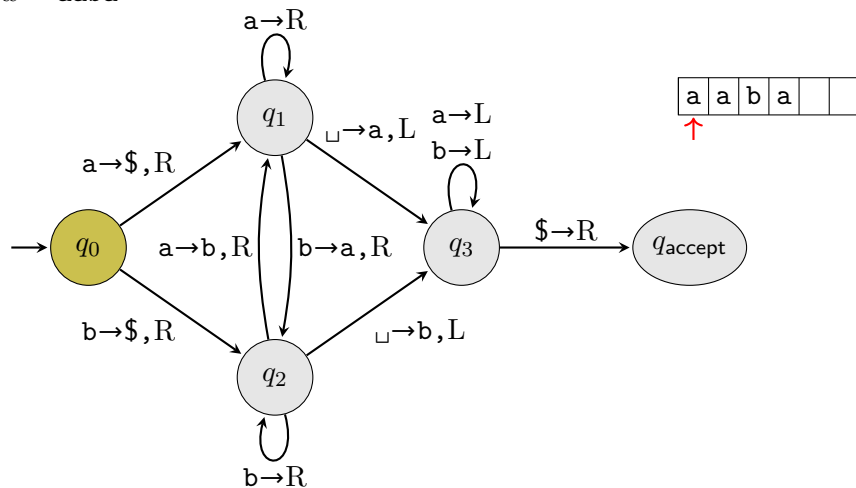




## Copying input right one cell

Consider  $\Sigma = \{a, b\}$  and a TM that copies its input one cell right, replacing the first cell with  $\$$  returns to the beginning and accepts

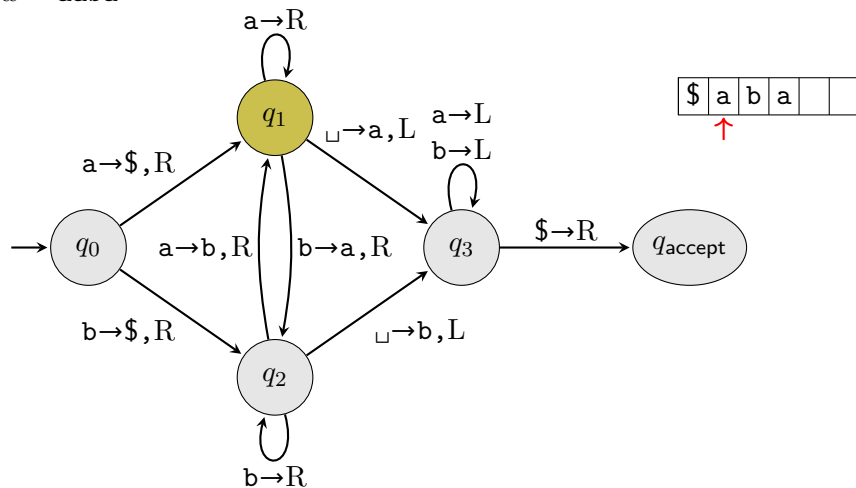
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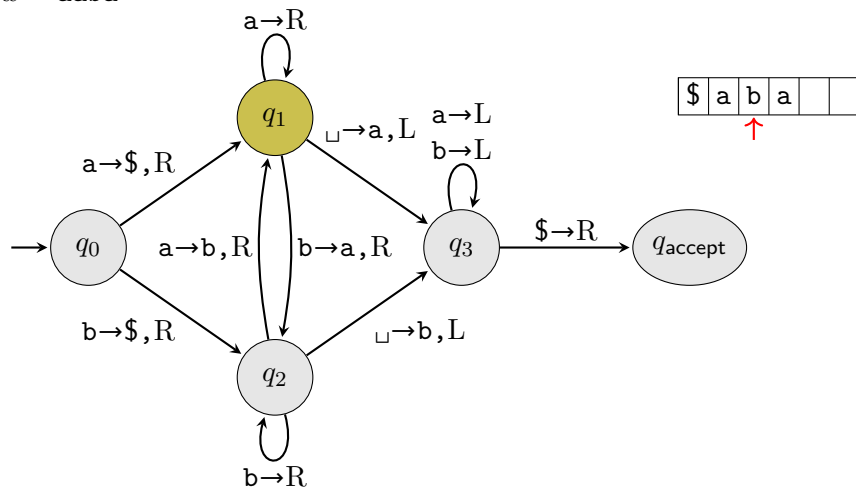
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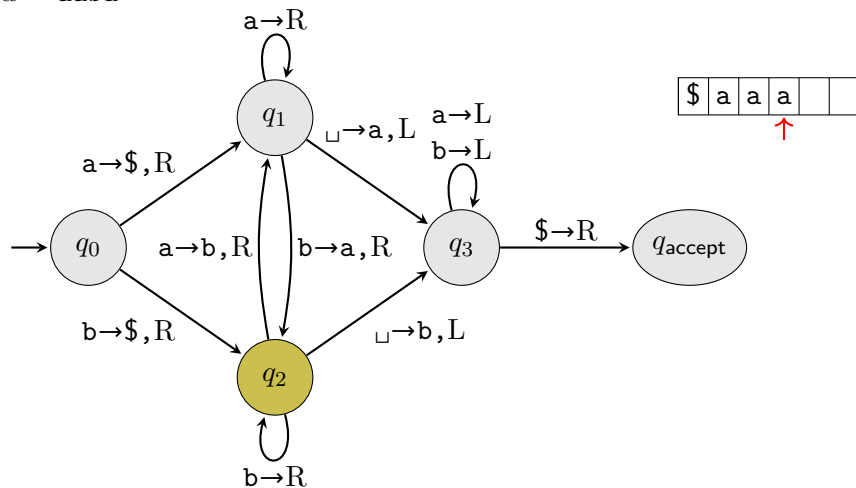
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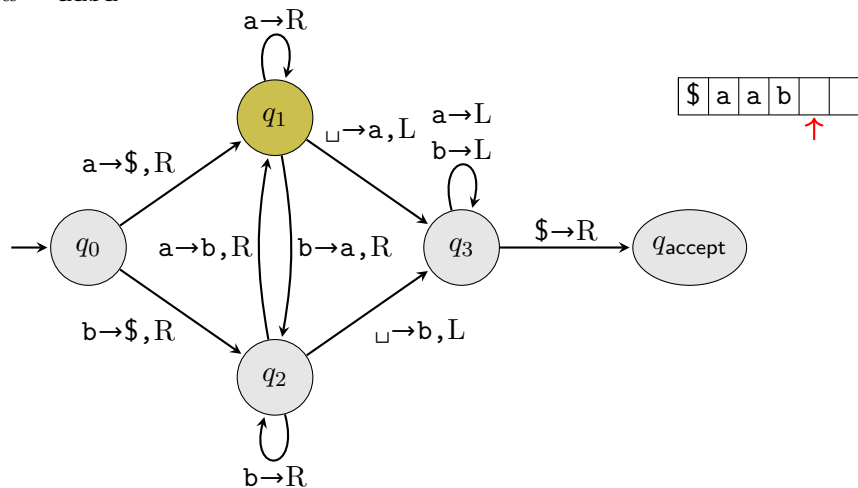
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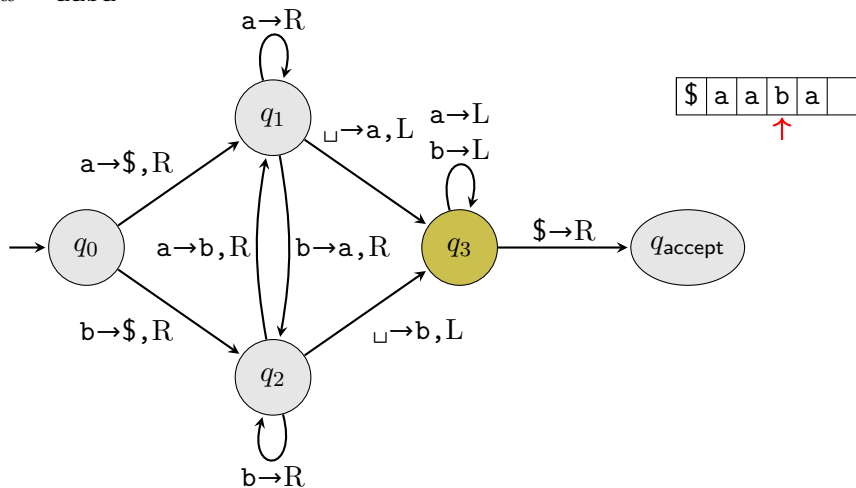
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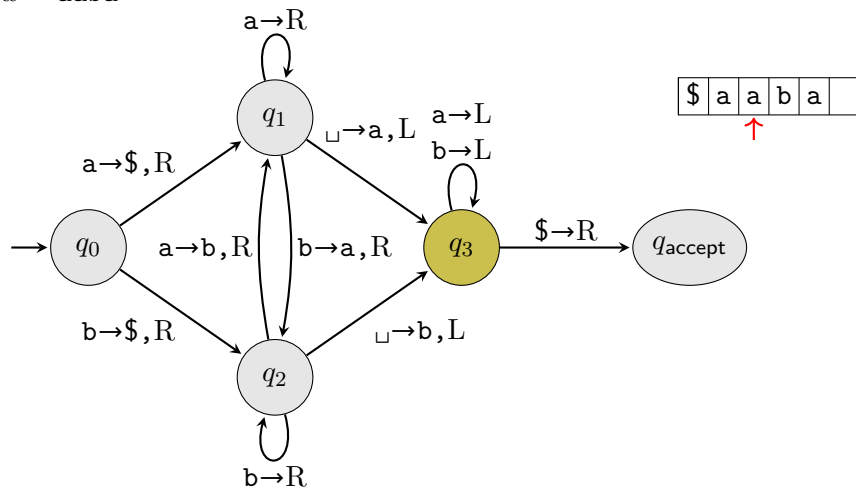
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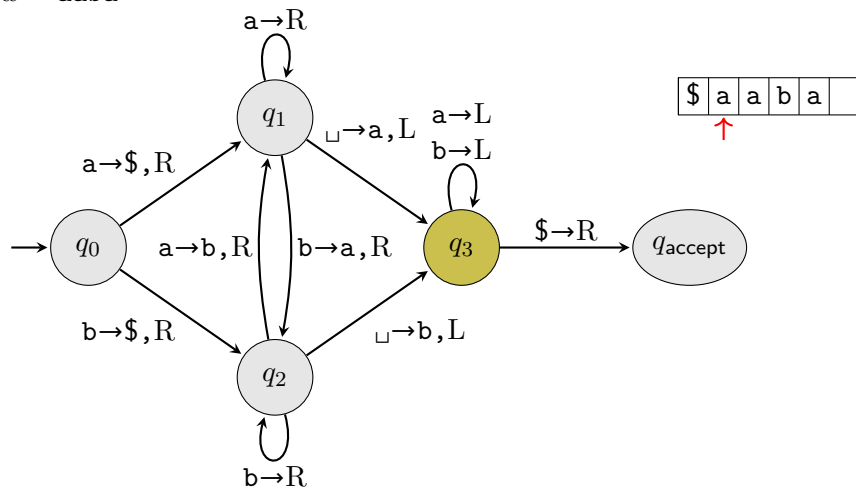
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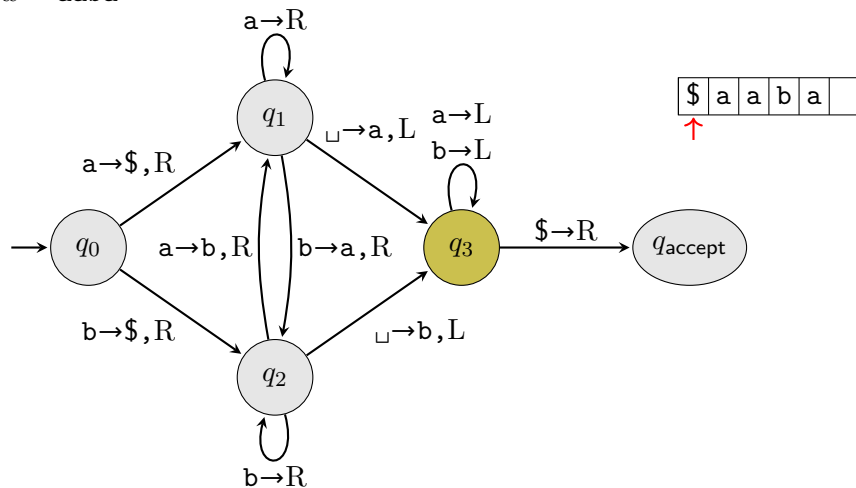




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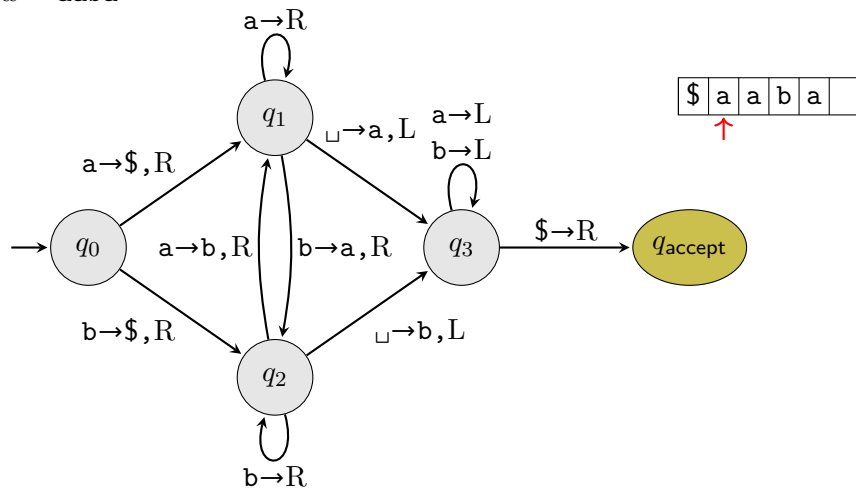
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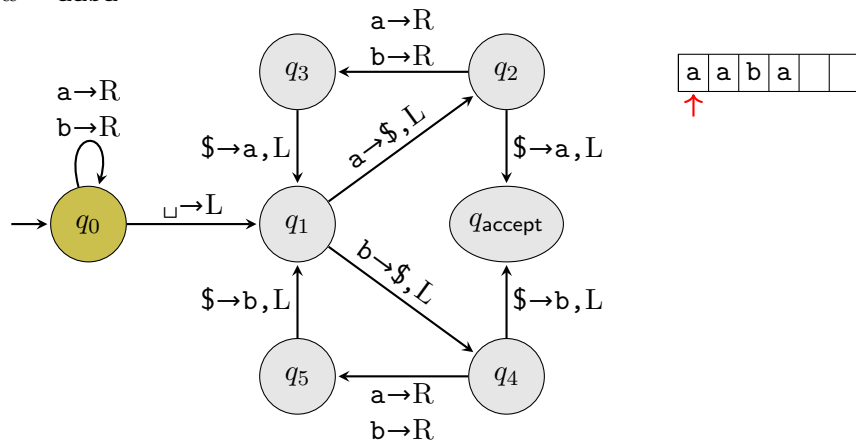
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## Returning to the beginning of the tape

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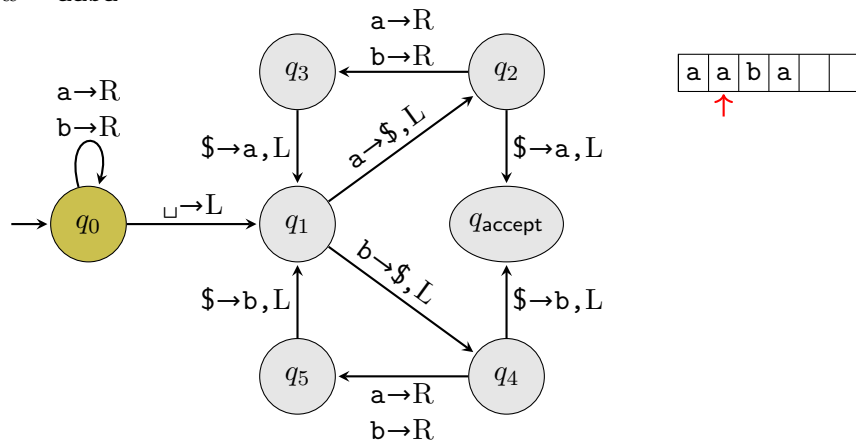
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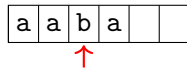
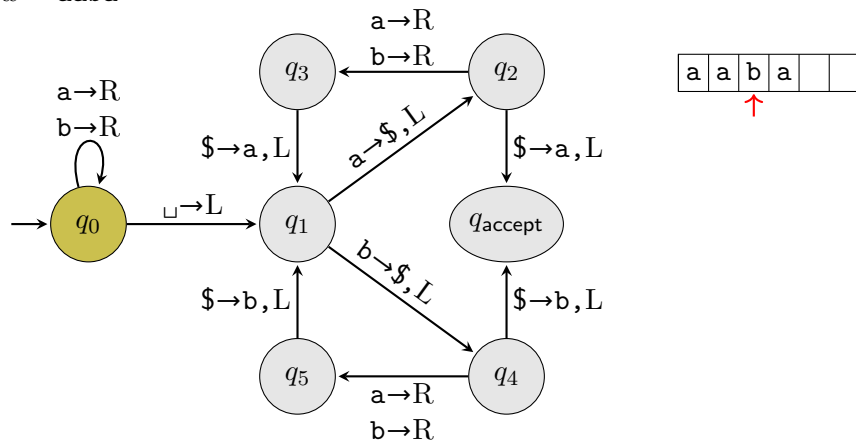
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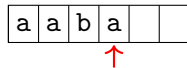
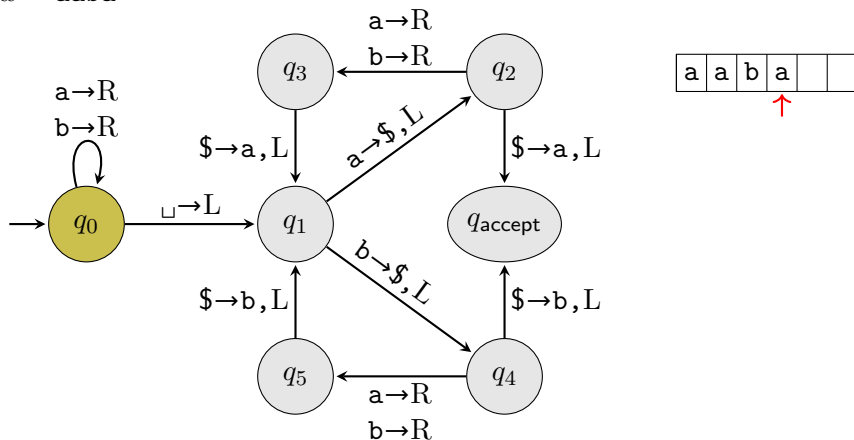
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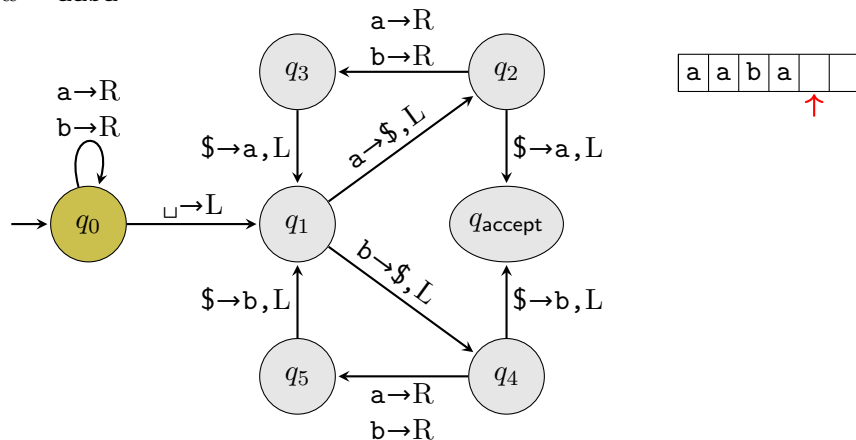
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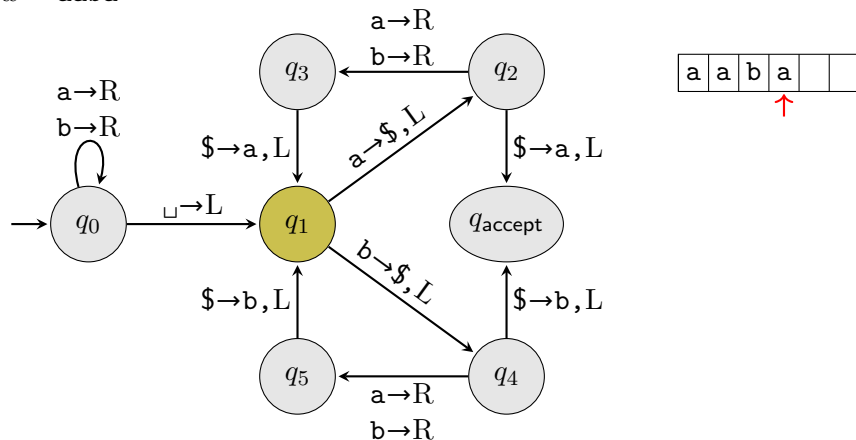
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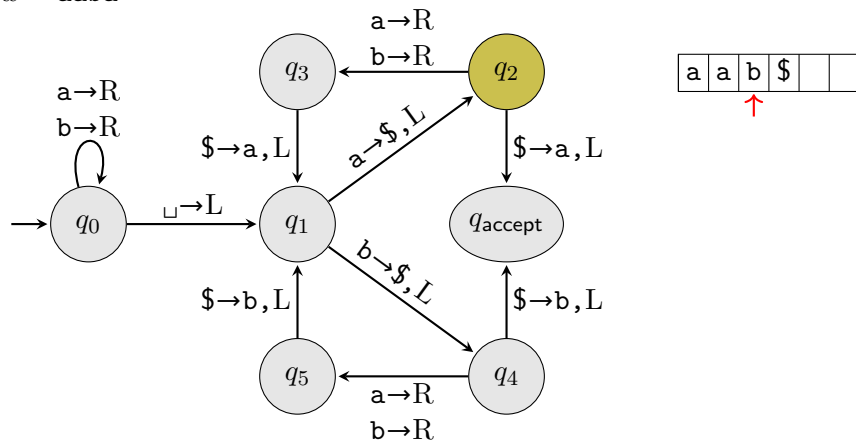




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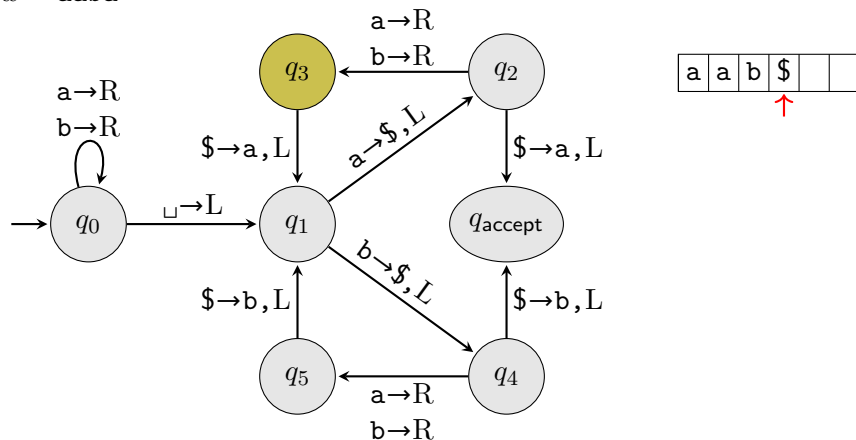
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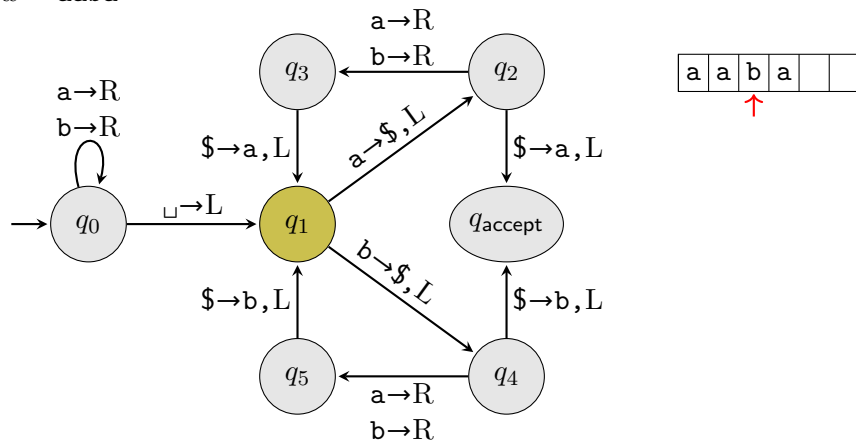
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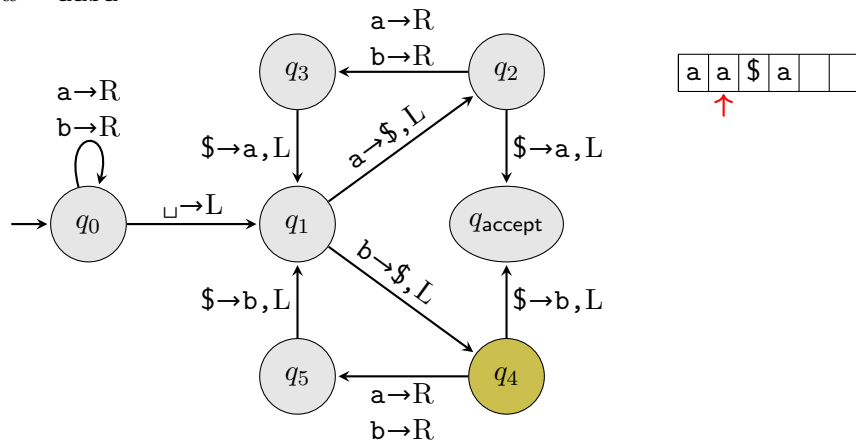
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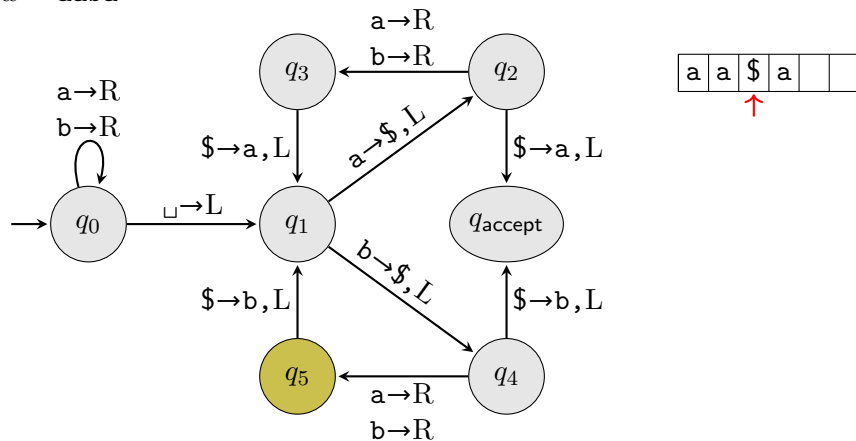
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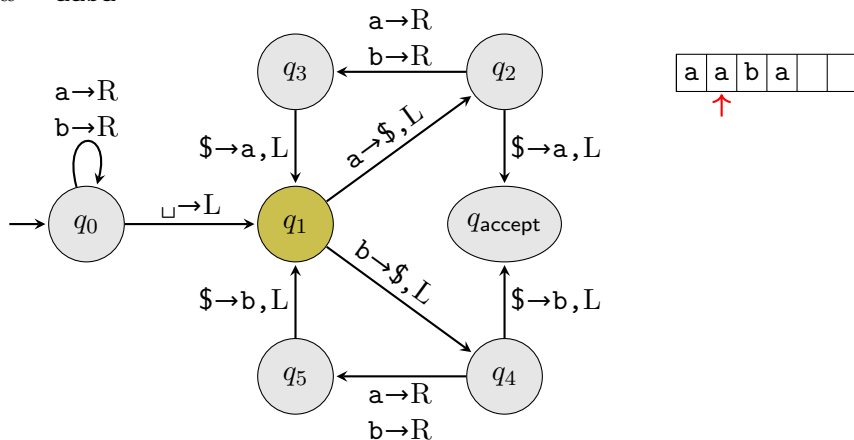
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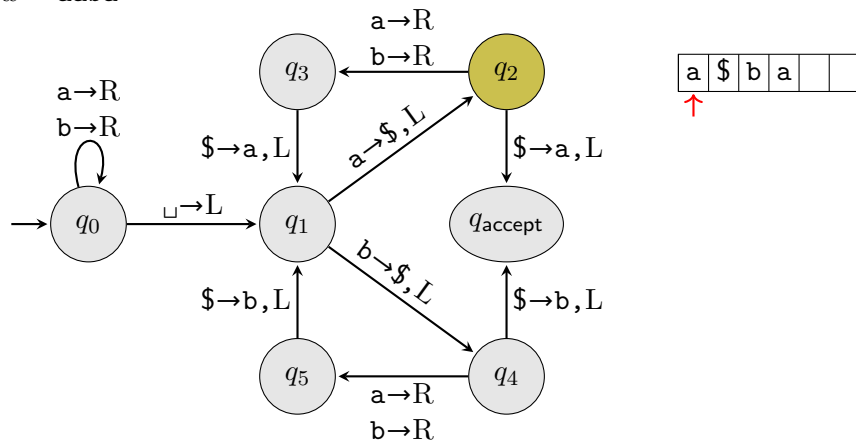
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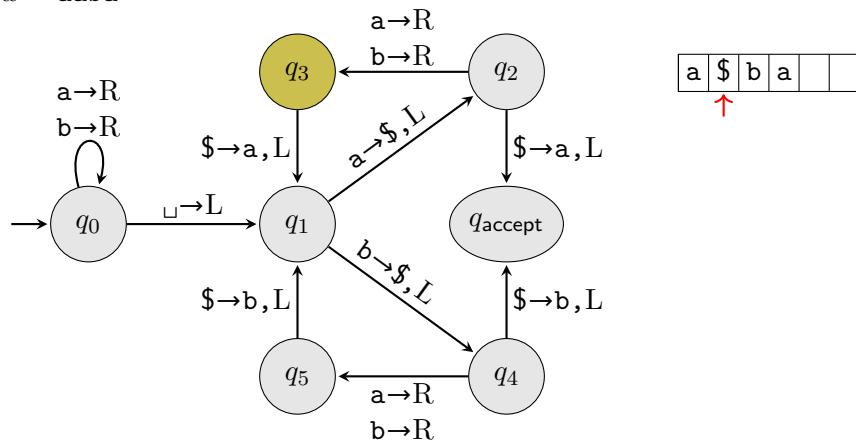
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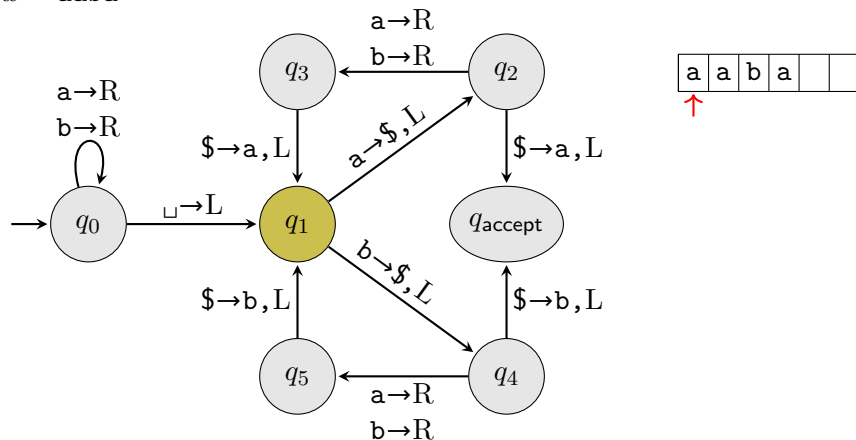




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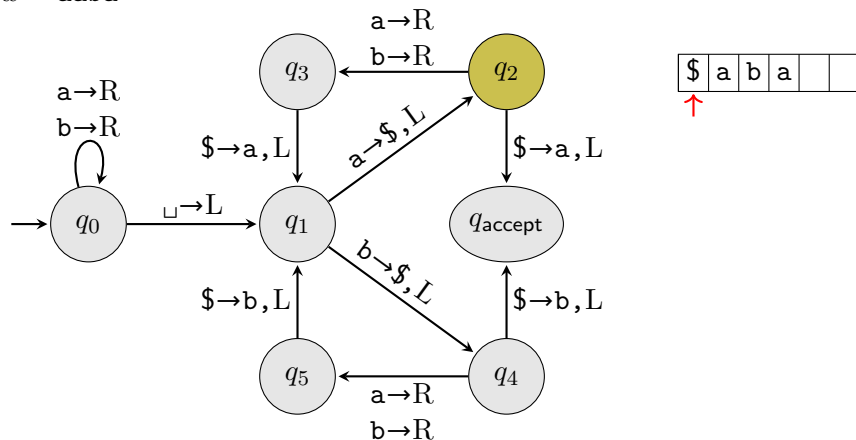
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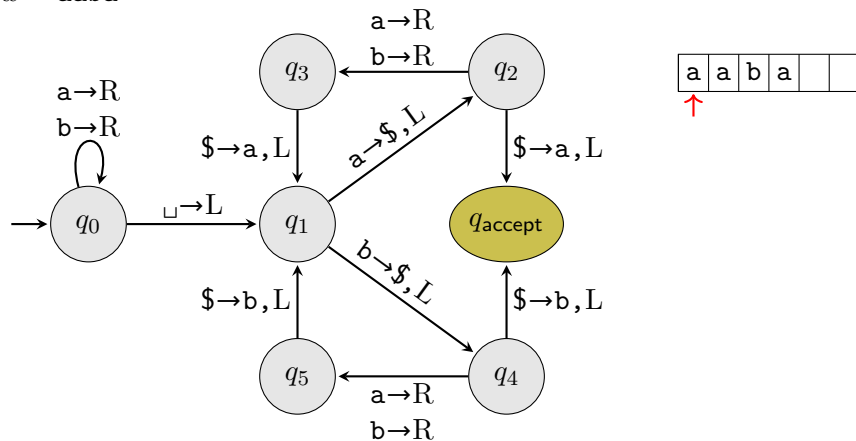
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## Equal number of as and bs

$C = \{w \mid w \in \{a, b\}^* \text{ has the same number of as and bs}\}$

$M =$  "On input  $w$ ,

- 1 Scan right and mark the first unmarked a, if none exist, goto step 4
- 2 Return the head to the beginning of the tape, scan right and mark the first unmarked b. If none exist, reject
- 3 Return the head to the beginning of the tape and goto step 1
- 4 Return the head to the beginning of the tape and scan right. If any unmarked bs exist, reject, otherwise accept."

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We can easily implement each of these steps using a few states in a TM, but this is much clearer

## What you *cannot* do

Each step in an implementation-level description (or a higher-level description) of a TM must be realizable in a real TM

You **cannot** have steps like

- “Check if  $w$  is in the language and accept, otherwise reject” (unless the language in question here is regular or other classes of languages we’ve shown are decidable)
- “If  $M$  loops, then .” There’s no way, in general, to determine if a TM is looping
- Equivalently, “If  $M$  doesn’t halt then .”

We’ll see more examples of things TM’s can’t do later