

CS 301

Lecture 14 – Non-context-free languages

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March 7, 2018



Review of “pumpable” languages

Recall we call a language L **pumpable** with pumping length p if for all $w \in L$ with $|w| \geq p$, there exist strings $x, y, z \in \Sigma^*$ with $w = xyz$ such that

- 1 for all $i \geq 0$, $xy^iz \in L$;
- 2 $|y| > 0$; and
- 3 $|xy| \leq p$

Then we proved that regular languages are pumpable

This let us prove a language was not regular by showing it isn't pumpable

CF-pumpability

A language L is **CF-pumpable** with pumping length p if for all $w \in L$ with $|w| \geq p$, there exist strings $u, v, x, y, z \in \Sigma^*$ such that

- 1 for all $i \geq 0$, $uv^i xy^i z \in L$;
- 2 $|vy| > 0$; and
- 3 $|vxy| \leq p$

Rather than dividing the string into 3 pieces, we're dividing it into 5

Two of the pieces (v and y) are pumped together

Condition 2 tells us that at least one of v or y must not be ε

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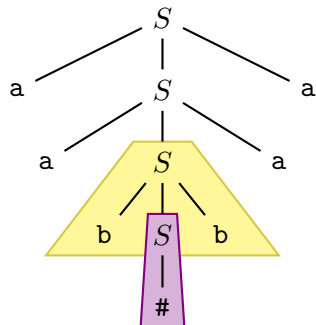
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- 2 $|vy| = |cc| = 2 > 0$
- 3 $|vxy| = |c\#c| = 3 \leq p$

Parse trees

CFG for $A = \{w\#w^R \mid w \in \{a, b\}^*\}$: $S \rightarrow aSa \mid bSb \mid \#$

Consider a parse tree for $w = aab\#baa$

$i = 1$:



$u = aa, v = b, x = \#, y = b, z = aa$

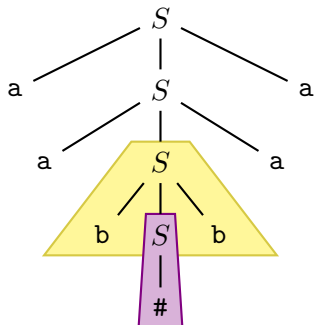
- Pumping down replaces the yellow trapezoid with the violet trapezoid
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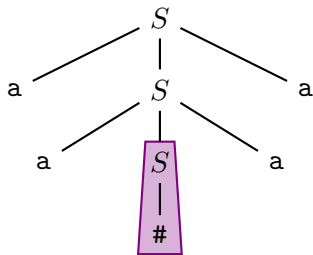
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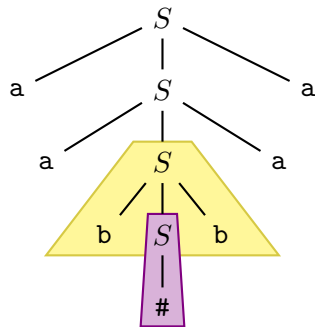
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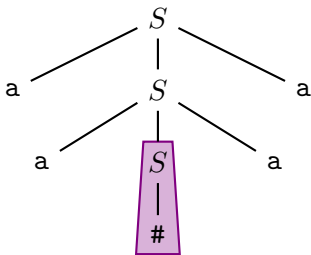
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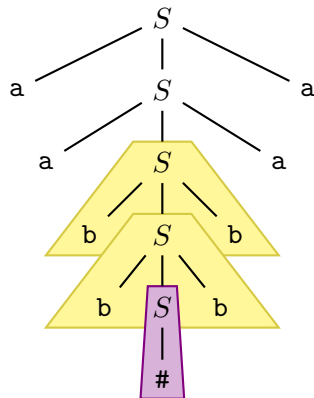
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Theorem (Pumping lemma for context-free languages)

Context-free languages are CF-pumpable

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Proof idea.

Consider a CFG $G = (V, \Sigma, R, S)$ in CNF

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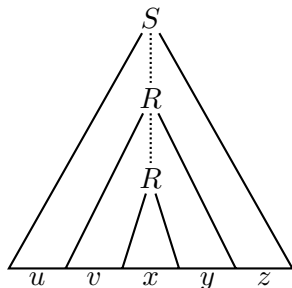
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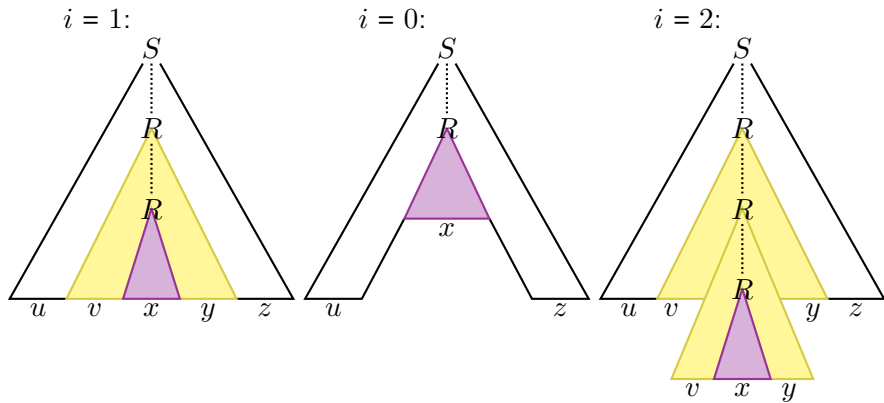
Set p large enough that any string of length at least p repeats some variable in its derivation (it turns out $p = 2^{|V|} + 1$ works)

This repeated variable, call it R , will play the same role as the repeated state did in proving that regular languages are pumpable

Note that this means $R \Rightarrow^* vxy$ and $R \Rightarrow^* x$



Condition 1: $\forall i \geq 0. uv^i xy^i z \in L$



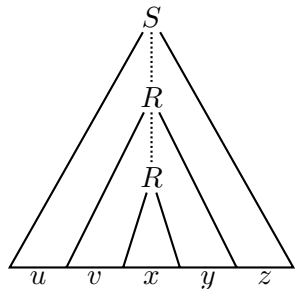
- Pumping down replaces the yellow triangle with the violet triangle
- Pumping up replaces the violet triangle with the yellow triangle
- We can pump up arbitrarily by repeating this process

Thus we've satisfied the first condition:

- 1 for all $i \geq 0, uv^i xy^i z \in L$

Condition 2: $|vy| > 0$

To see that at least one of v or y is not ε , let's look at $R \stackrel{*}{\Rightarrow} vRy$

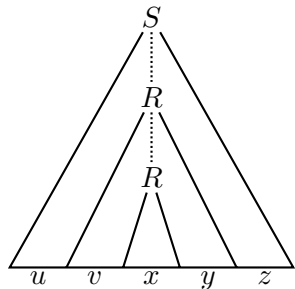


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Since G is in CNF, we must have $R \Rightarrow AB \stackrel{*}{\Rightarrow} vRy$ for some variables A and B

Two cases:



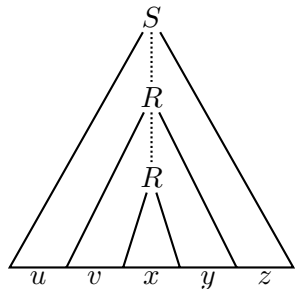
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- $A \stackrel{*}{\Rightarrow} vRs$ and $B \stackrel{*}{\Rightarrow} t$ where $st = y$
 t (and thus y) cannot be ε because G is in CNF



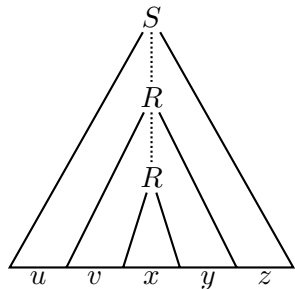
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 t (and thus y) cannot be ε because G is in CNF
- $A \stackrel{*}{\Rightarrow} s$ and $B \stackrel{*}{\Rightarrow} tRy$ where $st = v$
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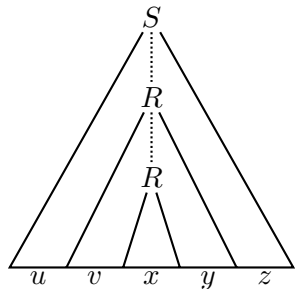
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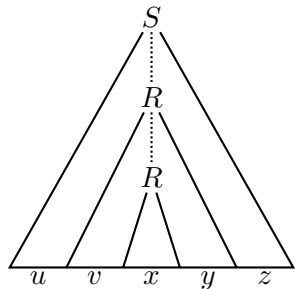
In either case, we've satisfied the second condition:

- ② $|vy| > 0$



Condition 3: $|vxy| \leq p$

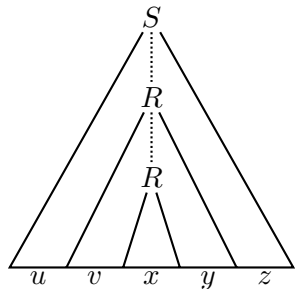
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Looking at all subtrees of height at most $|V| + 1$, there must be a repeated variable (pigeonhole principle), so pick one of those for R that derives vxy



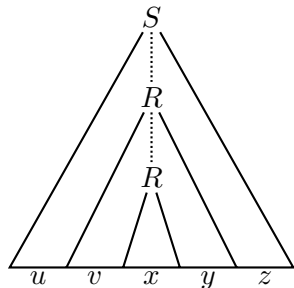
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Now since R is at distance at most $|V| + 1$ from the leaves, we must have $|vxy| \leq 2^{|V|} \leq p$

(A perfect binary tree of height h has 2^h leaves, but the last level of interior nodes in a parse tree for a grammar in CNF have a single child each)



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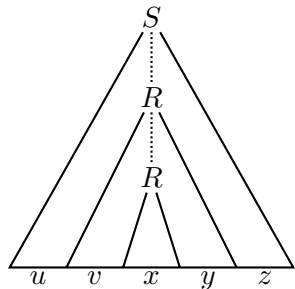
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Therefore, we've satisfied the final condition:

③ $|vxy| \leq p$



Showing that a language is not context-free

We can prove that a language is not context-free by showing that it violates the pumping lemma for context-free languages

Steps:

- 1 Assume the language, L , is context-free with some unspecified pumping length p
- 2 Pick string $w \in L$ such that $|w| \geq p$
- 3 Consider every division of w into $uvxyz = w$ such that $|vy| > 0$, and $|vxy| \leq p$
- 4 For each possible division, show that for some i , $uv^i xy^i z \notin L$

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- At least one of v or y contains two distinct symbols. Then uv^2xy^2z contains symbols out of order so $uv^2xy^2z \notin B$
- Both v and y contain the same symbol ($v = a^m, y = a^n$; $v = b^m, y = b^n$; or $v = c^m, y = c^n$). Then uxz doesn't have the same number of as, bs, and cs, so $uv^0xy^0z \notin B$

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- v and y contain different symbols, but only a single type each ($v = a^m, y = b^n$; $v = a^m, y = c^n$; or $v = b^m, y = c^n$). Again, uxz doesn't have the same number of as, bs, and cs so $uv^0xy^0z \notin B$

Using closure properties

Using the pumping lemma for CFLs is a *pain*

We can prove that

$$C = \{w \mid w \in \{a, b, c\}^* \text{ and } w \text{ has the same number of } a\text{s, } b\text{s, and } c\text{s}\}$$

is not context-free by intersecting it with a regular language

What language should we choose?

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$$C \cap \underline{a^*b^*c^*} = B$$

Since context-free languages are closed under intersection with a regular language, if C were context-free, then B would be context-free.

This is a contradiction so C is not context-free.

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- If x contains a b , then either $v = a^m$ is in the first run of a s and $y = a^n$ is in the second, or v is in the second and y is in the third. In either case, pumping down gives a string with a s in the wrong ratio

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- Try to cover as many similar cases at once as possible; e.g., if several cases are analogous, try to address them in one argument

Intersection of CFLs

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Is the intersection of two CFLs necessarily context-free?

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$$E = \{a^m b^m c^n \mid m, n \geq 0\}$$

$$F = \{a^m b^n c^n \mid m, n \geq 0\}$$

$$E \cap F = \{a^n b^n c^n \mid n \geq 0\}$$