CS 301

Lecture 07 – Closure properties of regular languages

Stephen Checkoway

February 7, 2018

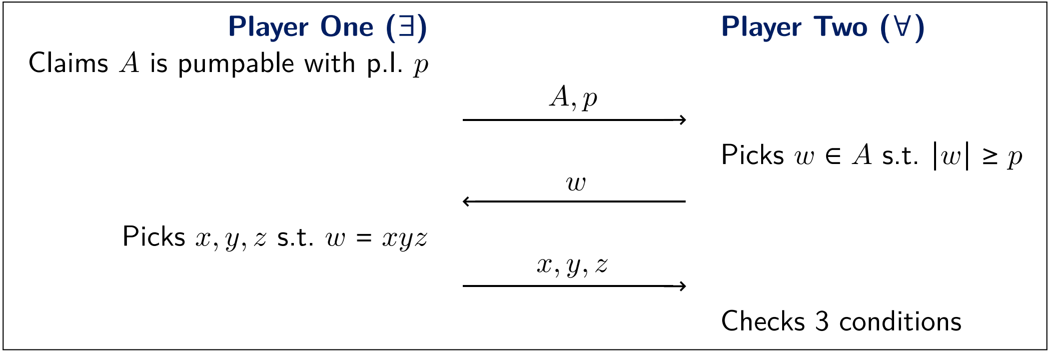
Last time: pumping lemma

Theorem

Pumping lemma for regular languages For every regular language A, there exists an integer $p > 0$ called the pumping length such that for every $w \in A$ there exist strings x, y, and z with $w = xyz$ such that

1. $xy^iz \in A$ *for all* $i \geq 0$
2. $|y| > 0$
3. $|xy| \leq p$.

A two-player game



Player One ($\exists$)

Claims A is pumpable with p.l. p

$\xrightarrow{A, p}$

Player Two ($\forall$)

Picks $w \in A$ s.t. $|w| \geq p$

$\xleftarrow{w}$

Picks x, y, z s.t. $w = xyz$

$\xrightarrow{x, y, z}$

Checks 3 conditions

Player One “wins” if

1. $xy^i z \in A$ for all $i \geq 0$
2. $|y| > 0$
3. $|xy| \leq p$

Can play as either Player One or Two

A two-player game

Player One ($\exists$)		**Player Two ($\forall$)**
Claims A is pumpable with p.l. p		
	$\xrightarrow{A, p}$	
		Picks $w \in A$ s.t. $\lvert w \rvert \geq p$
	$\xleftarrow{w}$	
Picks x, y, z s.t. $w = xyz$		
	$\xrightarrow{x, y, z}$	
		Checks 3 conditions

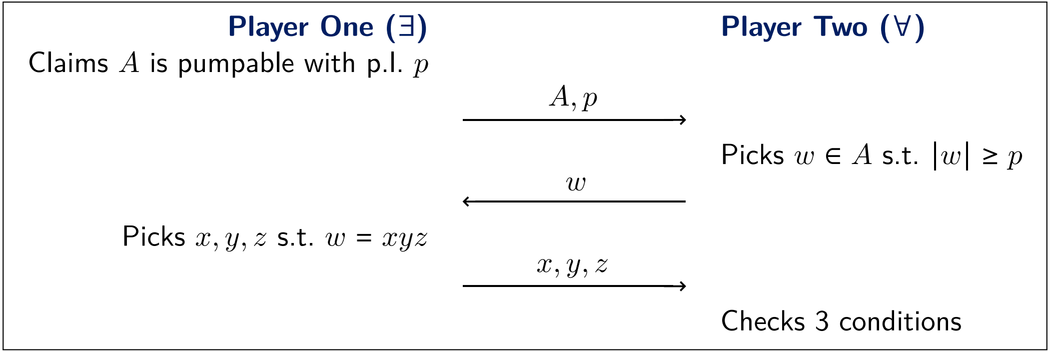
Player One “wins” if

1. $xy^i z \in A$ for all $i \geq 0$
2. $\lvert y \rvert > 0$
3. $\lvert xy \rvert \leq p$

Can play as either Player One or Two

- To show that A is pumpable, play as Player One
  You must consider all possible w and pick x, y, and z

A two-player game



Player One ($\exists$)

Claims A is pumpable with p.l. p

$\xrightarrow{A, p}$

Player Two ($\forall$)

Picks $w \in A$ s.t. $|w| \geq p$

$\xleftarrow{w}$

Picks x, y, z s.t. $w = xyz$

$\xrightarrow{x, y, z}$

Checks 3 conditions

Player One "wins" if

1. $xy^i z \in A$ for all $i \geq 0$
2. $|y| > 0$
3. $|xy| \leq p$

Can play as either Player One or Two

- To show that A is pumpable, play as Player One
  You must consider all possible w and pick x, y, and z
- To show that A is not pumpable, play as Player Two
  You must pick w and consider all possible x, y, and z

Last time: strategy for proving a language is not regular

To show that A is not regular, we assume it is and then find a string that cannot be "pumped"

Since we don't know the pumping length p, we have to construct a string w that depends on p

E.g., w might contain 0^p or $(\texttt{aba})^p$

Usually, we want to construct w such that the condition $|xy| \le p$ constrains the possible choices of x and y

Next, we consider all possible combination of x, y, and z such that $xyz = w$, $|xy| \le p$ and $|y| > 0$

Finally, for each combination, we find an $i \ge 0$ such that $xy^iz \notin A$

Duplicated strings

Prove that $A = \{xx \mid x \in \{0,1\}^*\}$ is not regular

Proof.

Assume A is regular with pumping length p.

What string should we pick?

Duplicated strings

Prove that $A = \{xx \mid x \in \{0,1\}^*\}$ is not regular

Proof.

Assume A is regular with pumping length p.

What string should we pick?

Let $w = 0^p 1^p 0^p 1^p$.

For $x, y, z \in \{0,1\}^*$ such that $xyz = w$, $|xy| \le p$, and $|y| > 0$,
we have $x = 0^m$, $y = 0^n$, and $z = 0^{p-m-n} 1^p 0^p 1^p$
Since $xy^0 z = 0^{p-n} 1^p 0^p 1^p \notin A$, A must not be regular. □

An easier method (sometimes)

Assume the language A is regular and apply closure properties of regular languages to arrive at a language that isn't regular

We know regular languages are closed under

- union
- concatenation
- Kleene star
- reversal
- complement
- intersection
- ...

If we, for example, intersect A with a regular language and end up with a nonregular language, then A is not regular

Same number of 0 and 1

Prove that $B = \{w \mid w \in \{0,1\}^* \text{ and } w \text{ has the same number of 0s as 1s}\}$ is not regular

Proof.

If B is regular, then $B \cap \underline{0^* 1^*} = \{0^n 1^n \mid n \geq 0\}$ is regular which is a contradiction so A must not be regular. ☐

Not duplicated strings

Prove that $C = \{xy \mid x, y \in \{0,1\}^*, |x| = |y|, \text{ and } x \neq y\}$ is not regular

Proof.

Assume C is regular.

Not duplicated strings

Prove that $C = \{xy \mid x, y \in \{0,1\}^*,\ |x| = |y|, \text{ and } x \neq y\}$ is not regular

Proof.

Assume C is regular.

Then $\overline{C} = \{xx \mid x \in \{0,1\}^*\} \cup \{y \mid |y| \text{ is odd}\}$ is regular.

Not duplicated strings

Prove that $C = \{xy \mid x, y \in \{0,1\}^*, |x| = |y|, \text{ and } x \neq y\}$ is not regular

Proof.

Assume C is regular.

Then $\overline{C} = \{xx \mid x \in \{0,1\}^*\} \cup \{y \mid |y| \text{ is odd}\}$ is regular.

Therefore $\overline{C} \cap \underline{(\Sigma\Sigma)^*} = \{xx \mid x \in \{0,1\}^*\}$ is regular

Not duplicated strings

Prove that $C = \{xy \mid x, y \in \{0,1\}^*, |x| = |y|, \text{ and } x \neq y\}$ is not regular

Proof.

Assume C is regular.

Then $\overline{C} = \{xx \mid x \in \{0,1\}^*\} \cup \{y \mid |y| \text{ is odd}\}$ is regular.

Therefore $\overline{C} \cap \underline{(\Sigma\Sigma)^*} = \{xx \mid x \in \{0,1\}^*\}$ is regular which is a contradiction so C must not be regular. □

A pumpable, nonregular language

$D = \{\mathtt{a}^k\mathtt{b}^m\mathtt{c}^n \mid \text{if } k = 1, \text{ then } m = n\}$ is pumpable with pumping length $p = 2$.

Consider a string $w = \mathtt{a}^k\mathtt{b}^m\mathtt{c}^n \in D$ with $|w| \geq 2$. We need to partition w into $xyz = w$ such that $xy^iz \in D$ for all $i \geq 0$, $|xy| \leq 2$, and $|y| > 0$

There are five cases to consider and in all of them, let $x = \varepsilon$

A pumpable, nonregular language

$D = \{\mathtt{a}^k\mathtt{b}^m\mathtt{c}^n \mid \text{if } k = 1\text{, then } m = n\}$ is pumpable with pumping length $p = 2$.

Consider a string $w = \mathtt{a}^k\mathtt{b}^m\mathtt{c}^n \in D$ with $|w| \geq 2$. We need to partition w into $xyz = w$ such that $xy^iz \in D$ for all $i \geq 0$, $|xy| \leq 2$, and $|y| > 0$

There are five cases to consider and in all of them, let $x = \varepsilon$

1. $k = 0$ and $m > 0$. Let $y = \mathtt{b}$ and $z = \mathtt{b}^{m-1}\mathtt{c}^n$. Thus $xy^iz = \mathtt{b}^{m+i-1}\mathtt{c}^n \in D$

A pumpable, nonregular language

$D = \{\mathtt{a}^k\mathtt{b}^m\mathtt{c}^n \mid \text{if } k = 1, \text{ then } m = n\}$ is pumpable with pumping length $p = 2$.

Consider a string $w = \mathtt{a}^k\mathtt{b}^m\mathtt{c}^n \in D$ with $|w| \geq 2$. We need to partition w into $xyz = w$ such that $xy^iz \in D$ for all $i \geq 0$, $|xy| \leq 2$, and $|y| > 0$

There are five cases to consider and in all of them, let $x = \varepsilon$

1. $k = 0$ and $m > 0$. Let $y = \mathtt{b}$ and $z = \mathtt{b}^{m-1}\mathtt{c}^n$. Thus $xy^iz = \mathtt{b}^{m+i-1}\mathtt{c}^n \in D$
2. $k = 0$ and $m = 0$. Let $y = \mathtt{c}$ and $z = \mathtt{c}^{n-1}$. Thus $xy^iz = \mathtt{c}^{n+i-1} \in D$

A pumpable, nonregular language

$D = \{\mathtt{a}^k\mathtt{b}^m\mathtt{c}^n \mid \text{if } k = 1, \text{ then } m = n\}$ is pumpable with pumping length $p = 2$.

Consider a string $w = \mathtt{a}^k\mathtt{b}^m\mathtt{c}^n \in D$ with $|w| \geq 2$. We need to partition w into $xyz = w$ such that $xy^iz \in D$ for all $i \geq 0$, $|xy| \leq 2$, and $|y| > 0$

There are five cases to consider and in all of them, let $x = \varepsilon$

1. $k = 0$ and $m > 0$. Let $y = \mathtt{b}$ and $z = \mathtt{b}^{m-1}\mathtt{c}^n$. Thus $xy^iz = \mathtt{b}^{m+i-1}\mathtt{c}^n \in D$
2. $k = 0$ and $m = 0$. Let $y = \mathtt{c}$ and $z = \mathtt{c}^{n-1}$. Thus $xy^iz = \mathtt{c}^{n+i-1} \in D$
3. $k = 1$. In this case, $m = n$. Let $y = \mathtt{a}$ and $z = \mathtt{b}^n\mathtt{c}^n$. Thus $xy^iz = \mathtt{a}^i\mathtt{b}^n\mathtt{c}^n \in D$

A pumpable, nonregular language

$D = \{\mathtt{a}^k\mathtt{b}^m\mathtt{c}^n \mid \text{if } k = 1\text{, then } m = n\}$ is pumpable with pumping length $p = 2$.

Consider a string $w = \mathtt{a}^k\mathtt{b}^m\mathtt{c}^n \in D$ with $|w| \ge 2$. We need to partition w into $xyz = w$ such that $xy^iz \in D$ for all $i \ge 0$, $|xy| \le 2$, and $|y| > 0$

There are five cases to consider and in all of them, let $x = \varepsilon$

1. $k = 0$ and $m > 0$. Let $y = \mathtt{b}$ and $z = \mathtt{b}^{m-1}\mathtt{c}^n$. Thus $xy^iz = \mathtt{b}^{m+i-1}\mathtt{c}^n \in D$
2. $k = 0$ and $m = 0$. Let $y = \mathtt{c}$ and $z = \mathtt{c}^{n-1}$. Thus $xy^iz = \mathtt{c}^{n+i-1} \in D$
3. $k = 1$. In this case, $m = n$. Let $y = \mathtt{a}$ and $z = \mathtt{b}^n\mathtt{c}^n$. Thus $xy^iz = \mathtt{a}^i\mathtt{b}^n\mathtt{c}^n \in D$
4. $k = 2$. Since m need not equal n, we need to be careful that pumping down doesn't leave us with one a. Let $y = \mathtt{aa}$ and $z = \mathtt{b}^m\mathtt{c}^n$. Thus $xy^iz = \mathtt{a}^{2i}\mathtt{b}^m\mathtt{c}^n \in D$

A pumpable, nonregular language

$D = \{\mathtt{a}^k\mathtt{b}^m\mathtt{c}^n \mid \text{if } k = 1, \text{ then } m = n\}$ is pumpable with pumping length $p = 2$.

Consider a string $w = \mathtt{a}^k\mathtt{b}^m\mathtt{c}^n \in D$ with $|w| \geq 2$. We need to partition w into $xyz = w$ such that $xy^iz \in D$ for all $i \geq 0$, $|xy| \leq 2$, and $|y| > 0$

There are five cases to consider and in all of them, let $x = \varepsilon$

1. $k = 0$ and $m > 0$. Let $y = \mathtt{b}$ and $z = \mathtt{b}^{m-1}\mathtt{c}^n$. Thus $xy^iz = \mathtt{b}^{m+i-1}\mathtt{c}^n \in D$
2. $k = 0$ and $m = 0$. Let $y = \mathtt{c}$ and $z = \mathtt{c}^{n-1}$. Thus $xy^iz = \mathtt{c}^{n+i-1} \in D$
3. $k = 1$. In this case, $m = n$. Let $y = \mathtt{a}$ and $z = \mathtt{b}^n\mathtt{c}^n$. Thus $xy^iz = \mathtt{a}^i\mathtt{b}^n\mathtt{c}^n \in D$
4. $k = 2$. Since m need not equal n, we need to be careful that pumping down doesn't leave us with one a. Let $y = \mathtt{aa}$ and $z = \mathtt{b}^m\mathtt{c}^n$. Thus $xy^iz = \mathtt{a}^{2i}\mathtt{b}^m\mathtt{c}^n \in D$
5. $k \geq 3$. Let $y = \mathtt{a}$ and $z = \mathtt{a}^{k-1}\mathtt{b}^m\mathtt{c}^n$. Then $xy^iz = \mathtt{a}^{k+i-1}\mathtt{b}^m\mathtt{c}^n$ Since $k \geq 3$, $k + i - 1 \geq 2$ so it doesn't matter if $m = n$ or not. Thus $xy^iz \in D$

In each case, $|xy| \leq 2$ and $|y| > 0$. Thus D is pumpable

A pumpable, nonregular language

$D = \{\mathtt{a}^k\mathtt{b}^m\mathtt{c}^n \mid \text{if } k = 1\text{, then } m = n\}$ is not regular

Proof.

Assume D is regular and intersect with $\underline{\mathtt{a}\mathtt{b}^*\mathtt{c}^*}$ giving the language $E = \{\mathtt{a}\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$.

By assumption D is regular so E is regular with pumping length p.

Let $w = \mathtt{a}\mathtt{b}^p\mathtt{c}^p$ and consider all partitions $xyz = w$ with $|xy| \leq p$ and $|y| > 0$.

If y contains $\mathtt{a}$, then xy^0z does not start with $\mathtt{a}$ so it's not in E

If y does not contain $\mathtt{a}$, then $x = \mathtt{a}\mathtt{b}^m$, $y = \mathtt{b}^n$, and $z = \mathtt{b}^{p-m-n}\mathtt{c}^p$ for some m and n. Therefore, $xy^0z = \mathtt{a}\mathtt{b}^{p-n}c^p \notin E$.

Therefore E is not regular so D must not be regular. $\square$

Unequal numbers of 0s and 1s

Let $F = \{\texttt{0}^m\texttt{1}^n \mid m \neq n\}$
Let $G = \{w \mid w \in \{\texttt{0},\texttt{1}\}^* \text{ and } w \text{ has an unequal number of 0s and 1s}\}$
Neither F nor G is regular

Easy proof via closure properties.

Note that $\overline{F} \cap \underline{\texttt{a}^*\texttt{b}^*} = \{\texttt{0}^n\texttt{1}^n \mid n \geq 0\}$ and $\overline{G} \cap \underline{\texttt{a}^*\texttt{b}^*} = \{\texttt{0}^n\texttt{1}^n \mid n \geq 0\}$. This is not regular so neither F nor G is regular. $\square$

Unequal numbers of 0s and 1s

Let $F = \{0^m 1^n \mid m \neq n\}$
Let $G = \{w \mid w \in \{0,1\}^*$ and w has an unequal number of 0s and 1s$\}$
Neither F nor G is regular

Hard proof via pumping lemma.

Assume F (resp. G) is regular with pumping length p.

What string w should we pick?

Unequal numbers of 0s and 1s

Let $F = \{0^m 1^n \mid m \neq n\}$
Let $G = \{w \mid w \in \{0,1\}^*$ and w has an unequal number of 0s and 1s$\}$
Neither F nor G is regular

Hard proof via pumping lemma.

Assume F (resp. G) is regular with pumping length p.

What string w should we pick?

Let $w = 0^p 1^{p+p!}$. Consider all partitions of $xyz = w$ such that $|xy| \le p$ and $|y| > 0$.
$x = 0^a$, $y = 0^b$, and $z = 0^{p-a-b} 1^{p+p!}$ for some $a \ge 0$ and $b > 0$

Unequal numbers of 0s and 1s

Let $F = \{0^m 1^n \mid m \neq n\}$
Let $G = \{w \mid w \in \{0,1\}^*$ and w has an unequal number of 0s and 1s$\}$
Neither F nor G is regular

Hard proof via pumping lemma.

Assume F (resp. G) is regular with pumping length p.

What string w should we pick?

Let $w = 0^p 1^{p+p!}$. Consider all partitions of $xyz = w$ such that $|xy| \leq p$ and $|y| > 0$.
$x = 0^a$, $y = 0^b$, and $z = 0^{p-a-b}1^{p+p!}$ for some $a \geq 0$ and $b > 0$

Set $i = p!/b + 1$ which is an integer because $b \leq p$ so b divides $p! = p \cdot (p-1) \cdots b \cdots 1$

$$\begin{aligned} xy^i z &= 0^{a+i\cdot b+(p-a-b)} 1^{p+p!} \\ &= 0^{a+(p!+b)+(p-a-b)} 1^{p+p!} \\ &= 0^{p+p!} 1^{p+p!} \end{aligned}$$

Since $xy^i z \notin F$, F is not regular (resp. $xy^i z \notin G$ so G is not regular) ☐

Complement and reversal of nonregular languages

Theorem

The class of nonregular languages is closed under complement and reversal.

Complement and reversal of nonregular languages

Theorem

The class of nonregular languages is closed under complement and reversal.

Proof.

Assume not. That is, assume language L is nonregular but $\overline{L}$ (resp. $L^{\mathcal{R}}$) is regular. Since $\overline{L}$ (resp. $L^{\mathcal{R}}$) is regular and regular languages are closed under complement (resp. reversal), $\overline{(\overline{L})} = L$ (resp. $(L^{\mathcal{R}})^{\mathcal{R}} = L$) is regular. This is a contradiction. ☐

Union, intersection, and star of nonregular languages

Theorem

The class of nonregular languages is not closed under union, intersection, or Kleene star

What steps would you take to prove these?

Union, intersection, and star of nonregular languages

Theorem

The class of nonregular languages is not closed under union, intersection, or Kleene star

What steps would you take to prove these?

Pick concrete, nonregular languages, apply the operation in question, and show that the result is regular

Union of nonregular languages

Proof that the class of nonregular languages is not closed under union.

Let $A = \{0^n 1^n \mid n \geq 0\}$. Since nonregular languages are closed under complement, $\overline{A}$ is nonregular.

Since $A \cup \overline{A} = \Sigma^*$ is regular, the class of nonregular languages is not closed under union. ☐

Question 1

Does this mean the union of any two nonregular languages is regular?

Question 1

Does this mean the union of any two nonregular languages is regular?

No. If A is nonregular, then $A \cup A = A$ is nonregular.

Operations more generally

$A \cup B$:

$A \setminus B$	Regular	Nonregular
Regular	Regular	Either
Nonregular	Either	Either

$A \cap B$:

$A \setminus B$	Regular	Nonregular
Regular	Regular	Either
Nonregular	Either	Either

$A \circ B$:

$A \setminus B$	Regular	Nonregular
Regular	Regular	Either
Nonregular	Either	Either

A^*:

A	
Regular	Regular
Nonregular	Either

$\overline{A}$:

A	
Regular	Regular
Nonregular	Nonregular

$A^{\mathcal{R}}$:

A	
Regular	Regular
Nonregular	Nonregular

It's worth spending time thinking up examples for the "Either" cases

Prefix, suffix, and quotient

For a language A over Σ, define

$$\textsc{Prefix}(A) = \{w \mid \text{for some } x \in \Sigma^*, \ wx \in A\}$$

$$\textsc{Suffix}(A) = \{w \mid \text{for some } x \in \Sigma^*, \ xw \in A\}$$

Prefix, suffix, and quotient

For a language A over Σ, define

$$\textsc{Prefix}(A) = \{w \mid \text{for some } x \in \Sigma^*,\ wx \in A\}$$
$$\textsc{Suffix}(A) = \{w \mid \text{for some } x \in \Sigma^*,\ xw \in A\}$$

For a string $u \in \Sigma^*$, define right and left quotient of A by u as

$$Au^{-1} = \{w \mid w \in \Sigma^* \text{ and } wu \in A\}$$
$$u^{-1}A = \{w \mid w \in \Sigma^* \text{ and } uw \in A\}$$

Prefix, suffix, and quotient

For a language A over Σ, define

$$\textsc{Prefix}(A) = \{w \mid \text{for some } x \in \Sigma^*,\ wx \in A\}$$
$$\textsc{Suffix}(A) = \{w \mid \text{for some } x \in \Sigma^*,\ xw \in A\}$$

For a string $u \in \Sigma^*$, define right and left quotient of A by u as

$$Au^{-1} = \{w \mid w \in \Sigma^* \text{ and } wu \in A\}$$
$$u^{-1}A = \{w \mid w \in \Sigma^* \text{ and } uw \in A\}$$

We can generalize to a quotient of A by a language B over Σ

$$A/B = \{w \mid \text{for some } x \in B,\ wx \in A\}$$
$$B\backslash A = \{w \mid \text{for some } x \in B,\ xw \in A\}$$

[Note, this is not $B \setminus A$]

Prefix, suffix, and quotient

Theorem

The class of regular languages is closed under $\textsc{Prefix}$, $\textsc{Suffix}$, *and quotient.*[1]

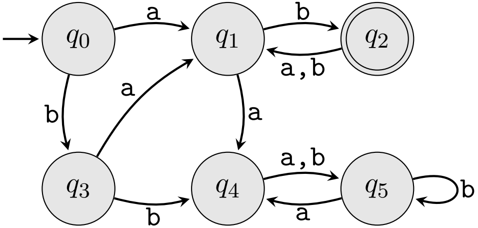
---

[1] We can make a stronger statement: If A is regular and B is *any* language, then A/B and $B\backslash A$ are regular.

Proof idea for closure under PREFIX

$\textsc{Prefix}(A) = \{w \mid \text{for some } x \in \Sigma^*, wx \in A\}$

Consider a DFA M



Some strings in $L(M)$

- ab
- bab
- abab
- babbb
- abbbab
- bababbb

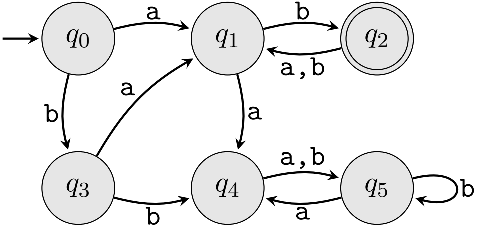
Some strings in $\textsc{Prefix}(L(M))$

- ε
- a
- b
- ab
- ba
- aba

Proof idea for closure under $\textsc{Prefix}$

$\textsc{Prefix}(A) = \{w \mid \text{for some } x \in \Sigma^*,\ wx \in A\}$

Consider a DFA M



Some strings in $L(M)$

- ab
- bab
- abab
- babbb
- abbbab
- bababbb

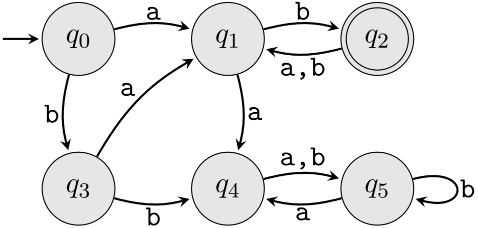
Some strings in $\textsc{Prefix}(L(M))$

- ε x = ab
- a
- b
- ab
- ba
- aba

Proof idea for closure under PREFIX

$\textsc{Prefix}(A) = \{w \mid \text{for some } x \in \Sigma^*, wx \in A\}$

Consider a DFA M



Some strings in $L(M)$

- ab
- bab
- abab
- babbb
- abbbab
- bababbb

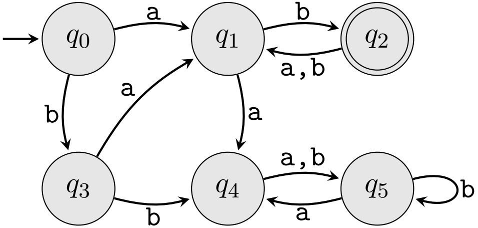
Some strings in $\textsc{Prefix}(L(M))$

- ε x = ab
- a x = b
- b
- ab
- ba
- aba

Proof idea for closure under Prefix

$\textsc{Prefix}(A) = \{w \mid \text{for some } x \in \Sigma^*,\ wx \in A\}$

Consider a DFA M



Some strings in $L(M)$

- ab
- bab
- abab
- babbb
- abbbab
- bababbb

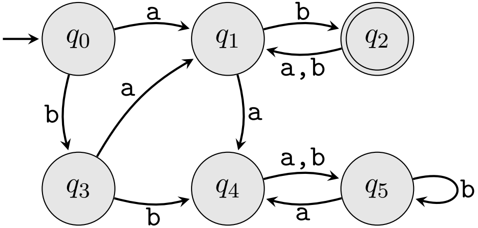
Some strings in $\textsc{Prefix}(L(M))$

- ε x = ab
- a x = b
- b x = ab
- ab
- ba
- aba

Proof idea for closure under Prefix

$\textsc{Prefix}(A) = \{w \mid \text{for some } x \in \Sigma^*, wx \in A\}$

Consider a DFA M



Some strings in $L(M)$

- ab
- bab
- abab
- babbb
- abbbab
- bababbb

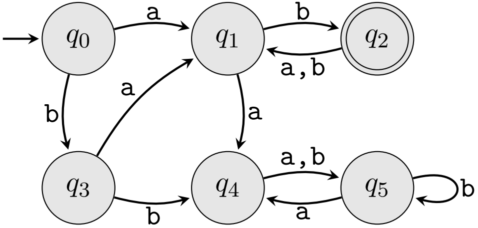
Some strings in $\textsc{Prefix}(L(M))$

- ε x = ab
- a x = b
- b x = ab
- ab $x = \varepsilon$
- ba
- aba

Proof idea for closure under $\textsc{Prefix}$

$\textsc{Prefix}(A) = \{w \mid \text{for some } x \in \Sigma^*, wx \in A\}$

Consider a DFA M



Some strings in $L(M)$

- ab
- bab
- abab
- babbb
- abbbab
- bababbb

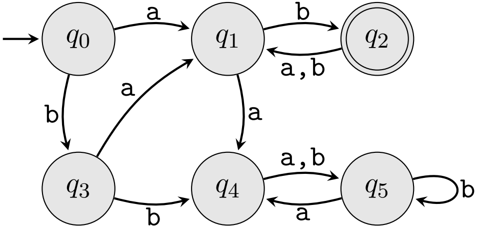
Some strings in $\textsc{Prefix}(L(M))$

- ε x = ab
- a x = b
- b x = ab
- ab $x = \varepsilon$
- ba x = b
- aba

Proof idea for closure under PREFIX

$\textsc{Prefix}(A) = \{w \mid \text{for some } x \in \Sigma^*, wx \in A\}$

Consider a DFA M



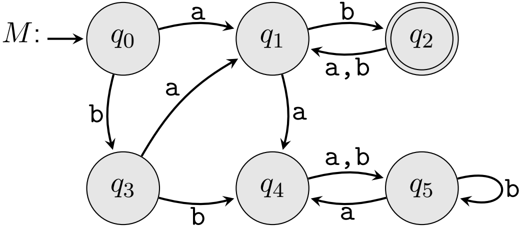
Some strings in $L(M)$

- ab
- bab
- abab
- babbb
- abbbab
- bababbb

Some strings in $\textsc{Prefix}(L(M))$

- ε $x =$ ab
- a $x =$ b
- b $x =$ ab
- ab $x = \varepsilon$
- ba $x =$ b
- aba $x =$ b

Proof idea for closure under PREFIX

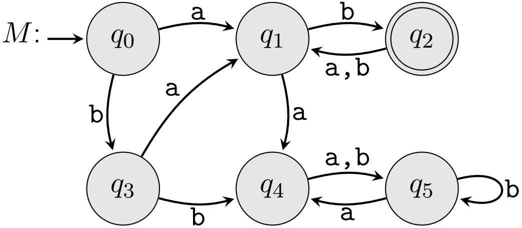


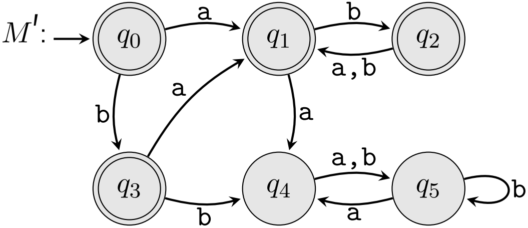
We want to build a new DFA M' s.t. $L(M') = \textsc{Prefix}(L(M))$

When M reads string w, it ends in some state q

w is a prefix of some string in $L(M)$ if there is some path through M from q to an accept state

Proof idea for closure under PREFIX





We want to build a new DFA M' s.t. $L(M') = \textsc{Prefix}(L(M))$

When M reads string w, it ends in some state q

w is a prefix of some string in $L(M)$ if there is some path through M from q to an accept state

This suggests a strategy: Build M' from M by making every state with a path to a state in F an accept state

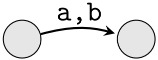
Regular languages are closed under PREFIX

Proof.

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA that recognizes A.

Construct $M' = (Q, \Sigma, \delta, q_0, F')$ where
$F' = \{q \mid q \in Q$ and there is a path from q to a state in $F\}$

---

[2] There may be multiple strings if one of the edges is labeled with multiple symbols 

Regular languages are closed under $\textsc{Prefix}$

Proof.

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA that recognizes A.

Construct $M' = (Q, \Sigma, \delta, q_0, F')$ where
$F' = \{q \mid q \in Q$ and there is a path from q to a state in $F\}$

Consider running M' on string w and ending in some state q

M' accepts $w \iff q \in F' \iff$ there is a path from q to some state in F. Let $x \in \Sigma^*$ be a string[2] corresponding to that path. Thus $wx \in A$

Therefore, M' accepts $w \iff w \in \textsc{Prefix}(A)$ so $\textsc{Prefix}(A)$ is regular. ☐

---

[2] There may be multiple strings if one of the edges is labeled with multiple symbols

Regular languages are closed under (left) quotient by a string

Proof.

We want to show that if A is a regular language over Σ and u is a string over Σ, then $u^{-1}A = \{x \mid x \in \Sigma^* \text{ and } ux \in A\}$ is regular

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA that recognizes A

We want to build an M' that acts on input x just like M does on input ux
How should we build M'?

Regular languages are closed under (left) quotient by a string

Proof.

We want to show that if A is a regular language over Σ and u is a string over Σ, then $u^{-1}A = \{x \mid x \in \Sigma^* \text{ and } ux \in A\}$ is regular

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA that recognizes A

We want to build an M' that acts on input x just like M does on input ux
How should we build M'?
Let q be the state M ends in after reading input u
Let $M' = (Q, \Sigma, \delta, q, F)$.

Regular languages are closed under (left) quotient by a string

Proof.

We want to show that if A is a regular language over Σ and u is a string over Σ, then $u^{-1}A = \{x \mid x \in \Sigma^* \text{ and } ux \in A\}$ is regular

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA that recognizes A

We want to build an M' that acts on input x just like M does on input ux
How should we build M'?
Let q be the state M ends in after reading input u
Let $M' = (Q, \Sigma, \delta, q, F)$.

If $r_0, r_1, \ldots, r_n$ are the states M goes through on input ux, then $r_{|u|}, r_{|u|+1}, \ldots, r_n$ are the states M' goes through on input x. Thus M' accepts $x \iff M$ accepts ux. $\square$

Another proof of nonregularity

$D = \{\mathtt{a}^k\mathtt{b}^m\mathtt{c}^n \mid \text{if } k = 1\text{, then } m = n\}$ is not regular

Proof.
$\mathtt{a}^{-1}(D \cap \underline{\mathtt{ab}^*\mathtt{c}^*}) = \{\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$ which is not regular but regular languages are closed under intersection and quotient so D must not be regular. □