

# **Programming Abstractions**

## **Lecture 11: Y Combinator**

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# How do we write a recursive function?

Easy, use `define`

```
(define len
  (λ (lst)
    (cond [(empty? lst) 0]
          [else (add1 (len (rest lst)))])))
```

For the rest of this lecture, we're not going to use `(define (fun args) ...)`

# How do we write a recursive function? (without using define)

Easy, use `letrec`

```
(letrec ([len  
        (λ (lst)  
          (cond [(empty? lst) 0]  
                [else (add1 (len (rest lst)))]))])  
  len)
```

Recall, this binds `len` to our function `(λ (lst) ...)` in the body of the `letrec`

This expression returns the procedure bound to `len` which computes the length of its argument

Why does this not work to create a length procedure? (Note `len` rather than `lenrec`.)

```
(let ([len
      (λ (lst)
        (cond [(empty? lst) 0]
              [else (add1 (len (rest lst)))]))]
      len)
```

- A. It would work but `lenrec` more clearly conveys the programmer's intent to write a recursive procedure
- B. `len` is not defined inside the  $\lambda$
- C. `len` is not defined in the last line
- D. `len` isn't being called in the last line, it's being returned and this is an error
- E. None of the above

# How do we write a recursive function?

(just using anonymous functions created via  $\lambda$ s)

Less easy, but let's give it a go!

```
( $\lambda$  (lst)
  (cond [(empty? lst) 0]
        [else (add1 (??? (rest lst)))]))
```

We need to put something in the recursive case in place of the ??? but what?

If we replace the ??? with

```
( $\lambda$  (lst) (error "List too long!"))
```

we'll get a function that correctly computes the length of empty lists, but fails with nonempty lists

# Put the **function itself** there?

```
(λ (lst)
  (cond [(empty? lst) 0]
        [else (add1 ((λ (lst)
                       (cond [(empty? lst) 0]
                             [else (add1 (??? (rest lst)))]))
                     (rest lst)))]))
```

Not a terrible attempt, we still have ???, but now we can compute lengths of the empty list and a single element list.

# Maybe we can abstract out the function

```
(λ (len)
  (λ (lst)
    (cond [(empty? lst) 0]
          [else (add1 (len (rest lst)))])))
```

This isn't a function that operates on lists!

It's a function that takes a function `len` as a parameter and returns a closure that takes a list `lst` as a parameter and computes a sort of length function using the passed in `len` function

# make-length

```
(define make-length
  (λ (len)
    (λ (lst)
      (cond [(empty? lst) 0]
            [else (add1 (len (rest lst)))])))
```

- This is the same function as before but bound to the identifier `make-length`
- ▶ The **orange text** (together with **purple text**) is the body of `make-length`
  - ▶ The **purple text** is the body of the closure returned by `(make-length len)`

```
(define L0 (make-length (λ (lst) (error "too long"))))
```

- ▶ `L0` correctly computes the length of the empty list but fails on longer lists

# make-length

```
(define make-length
  (λ (len)
    (λ (lst)
      (cond [(empty? lst) 0]
            [else (add1 (len (rest lst)))]))))

(define L0 (make-length (λ (lst) (error "too long"))))
(define L1 (make-length L0))
(define L2 (make-length L1))
(define L3 (make-length L2))
```

- ▶  $L_n$  correctly computes the length of lists of size at most  $n$
- ▶ We need an  $L_\infty$  in order to work for all lists
- ▶ `(make-length length)` would work correctly, but that's cheating!

# Enter the Y combinator

Y is a "fixed-point combinator"

▸ A combinator is a function that operates on functions (more or less)

If  $f$  is a function of one argument, then  $(Y f) = (f (Y f))$

```
(Y make-length)
=> (make-length (Y make-length))
=> (λ (lst)
    (cond [(empty? lst) 0]
          [else (add1 ((Y make-length) (rest lst)))]))
```

This is precisely the length function: `(define length (Y make-length))`

# How is (Y make-length) the same as length?

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=> (add1 (length '(2 3))) ; lst is bound to '(2 3)
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=> (add1 (add1 (add1 (cond [(empty? lst) 0] [...]))))
```

```
=> (add1 (add1 (add1 0)))
```

```
=> 3
```

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=> (add1 (add1 (add1 (cond [(empty? lst) 0] [...]))))
```

```
=> (add1 (add1 (add1 0)))
```

```
=> 3
```

# But wait, how can that work?

Two problems:

- ▶ We defined  $Y$  in terms of  $Y$ ! It's recursive and the whole point was to write recursive anonymous functions
- ▶  $(Y\ f) = (f\ (Y\ f))$  but then  
 $(f\ (Y\ f)) = (f\ (f\ (Y\ f))) = (f\ (f\ (f\ (Y\ f)))) = \dots$   
and this will never end

# Defining Y

```
(define Y
  (λ (f)
    ((λ (g) (f (g g)))
     (λ (g) (f (g g))))))
```

It's tricky to see what's going on but Y is a function of f and its body is applying the anonymous function  $(\lambda (g) (f (g g)))$  to the argument  $(\lambda (g) (f (g g)))$  and returning the result.

```
(Y foo) = ((λ (g) (foo (g g)))           ; By applying Y to foo
           (λ (g) (foo (g g))))
        = (foo ((λ (g) (foo (g g)))      ; By applying orange fun
                (λ (g) (foo (g g))))    ; to purple argument
        = (foo (Y foo))                 ; From definition of Y
```

# Never ending computation

This form of the Y-combinator doesn't work in Scheme because the computation would never end

We can fix this by using the related Z-combinator

```
(define z
  (λ (f)
    ((λ (g) (f (λ (v) ((g g) v))))
     (λ (g) (f (λ (v) ((g g) v)))))))
```

This is the argument to our recursive function

With this definition, we can create a length function

```
(define length (z make-length))
```

# What is length actually defined as here?

```
(define Z
  (λ (f)
    ((λ (g) (f (λ (v) ((g g) v))))
     (λ (g) (f (λ (v) ((g g) v)))))))
```

```
(define length (Z make-length))
```

```
(Z make-length)
```

```
=> ((λ (g) (make-length (λ (v) ((g g) v))))
```

```
    (λ (g) (make-length (λ (v) ((g g) v))))))
```

```
=> (make-length (λ (v) ((λ (g) (make-length (λ (v) ((g g) v))))
                        (λ (g) (make-length (λ (v) ((g g) v))))))
    v)))
```

# Let's apply some equivalences

```
(make-length (λ (v) ((λ (g) (make-length (λ (v) ((g g) v))))  
                    (λ (g) (make-length (λ (v) ((g g) v))))))  
            v)))
```

```
=> (make-length (λ (v) ((Z make-length) v)))
```

```
=> (cond [(empty? lst) 0]  
        [else (add1 ((λ (v) ((Z make-length) v))  
                    (rest lst)))]
```

```
=> (cond [(empty? lst) 0]  
        [else (add1 ((λ (v) (length v))  
                    (rest lst)))])
```

```
=> (cond [(empty? lst) 0]  
        [else (add1 (length (rest lst)))])
```

# We can use Z to make recursive functions

Given a recursive function of one variable

```
(define foo  
  (λ (x) ... (foo ...) ...))
```

we can construct this only using anonymous functions by way of Z

```
(Z (λ (foo) (λ (x) ... (foo ...) ...)))
```

Factorial

```
(Z (λ (fact)  
  (λ (n)  
    (if (zero? n)  
        1  
        (* n (fact (sub1 n)))))))
```

# Step by step

1. Write your recursive function normally with recursive calls:

```
(define foo ( $\lambda$  (x) ...))
```

2. Wrap the lambda in another, single-argument lambda whose argument has the same name as your function:

```
(define foo ( $\lambda$  (foo) ( $\lambda$  (x) ...)))
```

3. Apply Z to that

```
(define foo (Z ( $\lambda$  (foo) ( $\lambda$  (x) ...))))
```

4. Be thankful that programming language designers give us easier ways to write recursive functions!

Imagine a version of Scheme without `define` or `letrec`, how can we write a recursive function `foo` and call it on a list? In other words, how do we write

```
(letrec ([foo (λ (lst) (... (foo ...) ...))]
         (foo '(1 2 3))
```

A. 

```
(let ([foo (z (λ (lst)
                (... (foo ...) ...))]
         (foo '(1 2 3))
```

B. 

```
(let ([foo (z (λ (foo)
                (λ (lst)
                  (... (foo ...) ...))))]
         (foo '(1 2 3))
```

C. It's not possible to write recursive functions without `define` or `letrec` in Scheme

# What about multi-argument functions?

We can use apply!

```
(define z*  
  (λ (f)  
    ((λ (g) (f (λ args (apply (g g) args))))  
     (λ (g) (f (λ args (apply (g g) args)))))))
```

This is the list of arguments to our recursive function

# Example: map

```
( (z* (λ (map)
      (λ (proc lst)
        (cond [(empty? lst) empty]
              [else (cons (proc (first lst))
                          (map proc (rest lst)))])))
  add1
  '(1 2 3 4 5))
```

We're applying  $z^*$  to the **orange function** which returns a recursive map procedure

Then we're applying that procedure to the arguments `add1` and `'(1 2 3 4 5)`