CS 271: Automata and Computation Theory **Spring 2015**

Problem Set #2

Due: Monday, March 2, 2014

Problem 1 Give a CFG that generates each of the following languages. For each variable in your CFG, describe the set of strings generated by that variable.

a. $\{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i = j \text{ or } j = k\}$

b. $\{x_1\#y_1\#x_2\#y_2\#\cdots\#x_n\#y_n \mid n > 0 \text{ and } x_i^{\mathcal{R}} \text{ is a substring of } y_i \text{ for each } i\}$ where $x_i, y_i \in \{\mathtt{a},\mathtt{b},\mathtt{c}\}^*$. [*Hint:* $y_i = \{\mathtt{a},\mathtt{b},\mathtt{c}\}^* x_i^{\mathcal{R}} \{\mathtt{a},\mathtt{b},\mathtt{c}\}^*$. *Let one variable generate* $\{\mathtt{a},\mathtt{b},\mathtt{c}\}^*$, *let another generate* $x\#y$ *where* $x^{\mathcal{R}}$ *is a substring of* y.]

Problem 2 Prove that every regular language is context-free by using the fact that CFLs are closed under union, concatenation, and Kleene star and that every regular language is generated by a regular expression. [*Hint: There are 6 cases to consider.*]

Problem 3 We proved that the language $\{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid \text{if } i = 1, \text{ then } j = k\}$ is not regular. Show that it is context-free by using closure properties of CFLs to construct it from simpler languages.

Problem 4 Convert the following CFG into CNF. You may use either the procedure in the book or the procedure discussed in class. Show each step.

$$S \to TST \mid T \mid \varepsilon$$
$$T \to \mathtt{a}T\mathtt{b} \mid \varepsilon$$

Problem 5 Prove that the class of context-free languages is closed under reversal. [*Hint: Consider a CFG in CNF and use induction on the length of the strings.*]

Problem 6 Prove that the class of context-free languages is closed under homormophism. [*Hint: To simplify the notation, consider a CFG that's in CNF and construct one that isn't necessarily in CNF.*]

Problem 7 Use the result from Problem 6 to show that $L = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n\mathtt{d}^n \mid n \geqslant 0\}$ is not context-free.

Problem 8 Prove that the following CFG generates the language

$$\{xy \mid x, y \in \Sigma^*,\ |x| = |y|,\ \text{and } x \neq y\}.$$

$$
\begin{aligned}
S &\to AB \mid BA \\
A &\to XAX \mid \texttt{a} \\
B &\to XBX \mid \texttt{b} \\
X &\to \texttt{a} \mid \texttt{b}
\end{aligned}
$$

[*Hint: Consider* m *applications of the rule* $A \to XAX$ *and* n *applications of the rule* $B \to XBX$ *in the derivation of a string and note that each instance of an* X *gives you a single terminal. So*

$$S \Rightarrow AB \stackrel{*}{\Rightarrow} X^m\texttt{a}X^mB \stackrel{*}{\Rightarrow} X^m\texttt{a}X^mX^n\texttt{b}X^n$$

where

$$X^i = \underbrace{XX\cdots X}_{i}.$$

All strings derived from the rightmost expression have length $2(m+n+1)$. *Now divide such a string into two* $m+n+1$ *parts and show that the two parts differ in at least one position. There's a similar argument when the first step in the derivation is* $S \Rightarrow BA$.]

Problem 9 We have used the fact that a CFG in CNF derives a string w of length $|w| = n > 0$ in exactly $2n-1$ steps. Prove this fact.

Problem 10 Show that the language $\{w \mid w \text{ has an equal number of } \texttt{a}\text{s}, \texttt{b}\text{s}, \text{and } \texttt{c}\text{s}\}$ is not context-free. [*Hint: The pumping lemma for CFLs is* not *the easiest way to do this.*]