CS 383

Lecture 15 – Turing machines

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Fall 2023

Machine models of computation

- DFA: Deterministic, finite set of states, no additional memory

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- PDA: Nondeterministic, finite set of states, infinite stack

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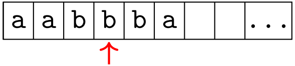
- DFA: Deterministic, finite set of states, no additional memory
- NFA: Nondeterministic, finite set of states, no additional memory
- PDA: Nondeterministic, finite set of states, infinite stack
- Turing machine (TM): Deterministic, finite set of states, one-way infinite, read-write “tape”

Turing machine

Model of computation as powerful as a computer

- Finite set of states (the control)
- Initial state
- Special accept and reject states (must be different!)
- One-way infinite tape divided into cells
- Tape head points at the current cell of the tape; can move left and right

Tape:



a	a	b	b	b	a			. . .

Single step

One step of computation

1. Read the symbol under the tape head
2. Based on the current state and the symbol read,
    1. Write a symbol to the tape (it can be the same symbol just read)
    2. Transition to a new state
    3. Move the tape head left or right (in some formulations, we allow the head to stay in place)

If the tape head is at the left end of the tape and the TM tries to move left, the tape head stays in place

TM Computation

TM computation

1. Start with

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   - the input w at the left end of the tape and blank symbols (which we indicate with $\sqcup$) extend infinitely to the right of the input;

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   - the tape head on the left-most cell of the tape (on the first symbol of the input if $w \neq \varepsilon$); and

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   - in the initial state

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2. If the current state is the accept state, halt and accept

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4. Perform one step of computation

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5. Goto step 2

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2. If the current state is the accept state, halt and accept
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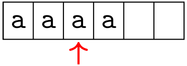
4. Perform one step of computation
5. Goto step 2

Three possible outcomes: accept, reject, loop (run forever)

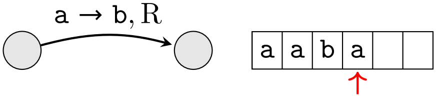
TM transitions

Starting with the tape



Transitions:

- Replace a with b and move right



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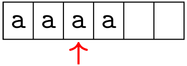

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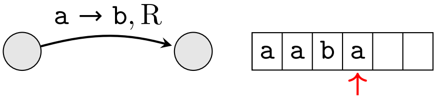

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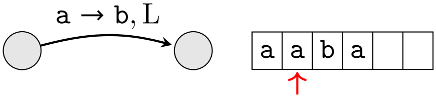

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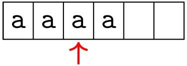

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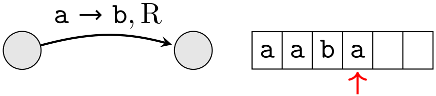

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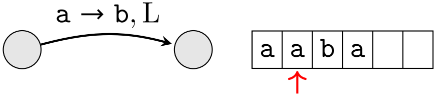
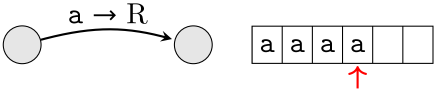
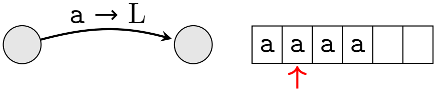


When the tape cell doesn't change, we use a shorthand

- On a, move right



- On a, move left

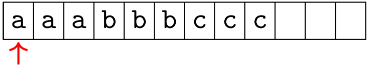


Example

Consider building a TM to recognize the language $A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

Let's start by thinking about the algorithm we would like the TM to perform

1. The TM starts with the input (say w = aaabbbccc is our input) on the tape



a	a	a	b	b	b	c	c	c			

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1. The TM starts with the input (say w = `aaabbbccc` is our input) on the tape

a	a	a	b	b	b	c	c	c			
↑											

2. Replace the `a` with an `x` and then scan right to a `b`

x	a	a	b	b	b	c	c	c			
			↑								

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a	a	a	b	b	b	c	c	c			
↑											

2. Replace the a with an x and then scan right to a b

x	a	a	b	b	b	c	c	c			
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3. Replace the b with a y and scan right to a c

x	a	a	y	b	b	c	c	c			
						↑					

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↑											

2. Replace the a with an x and then scan right to a b

x	a	a	b	b	b	c	c	c			
			↑								

3. Replace the b with a y and scan right to a c

x	a	a	y	b	b	c	c	c			
						↑					

4. Replace the c with a z and scan left until an x

x	a	a	y	b	b	z	c	c			
↑											

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↑											

2. Replace the a with an x and then scan right to a b

x	a	a	b	b	b	c	c	c			
			↑								

3. Replace the b with a y and scan right to a c

x	a	a	y	b	b	c	c	c			
						↑					

4. Replace the c with a z and scan left until an x

x	a	a	y	b	b	z	c	c			
↑											

5. Move right

x	a	a	y	b	b	z	c	c			
	↑										

Example

6. If the cell under the tape head is a, goto step 1

Example

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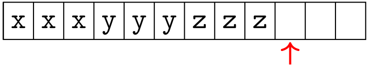
7. Otherwise, scan right through all x, y, and z



x	x	x	y	y	y	z	z	z			

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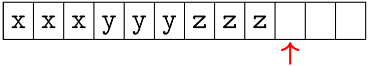

x	x	x	y	y	y	z	z	z			

8. If the cell under the tape head is ␣, accept

If at any point in the process the TM encounters a symbol it isn't expecting, reject

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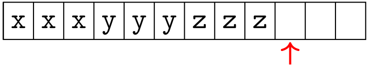


x	x	x	y	y	y	z	z	z			

8. If the cell under the tape head is ␣, accept

If at any point in the process the TM encounters a symbol it isn't expecting, reject

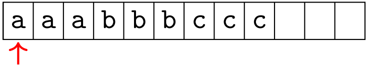
What should the TM do if the very first symbol is ␣?

It should accept because the string $\varepsilon \in A$

Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$



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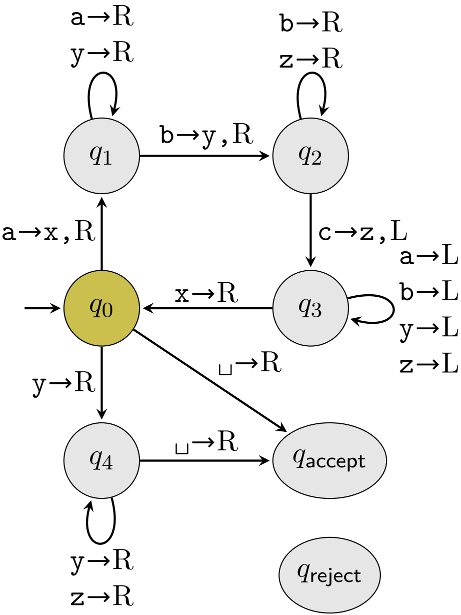

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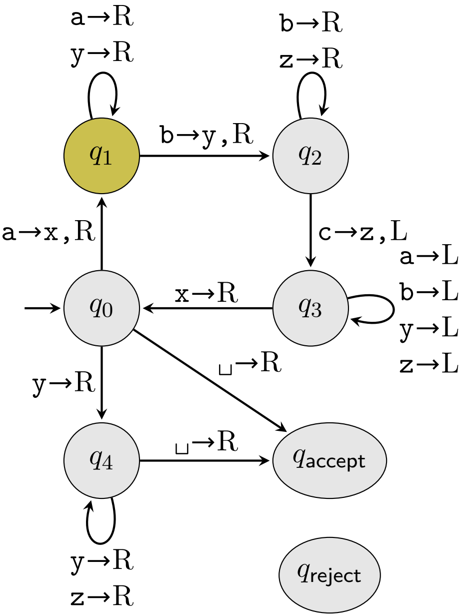

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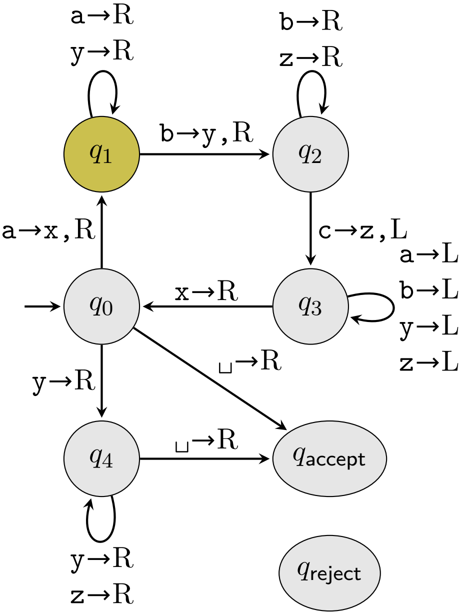

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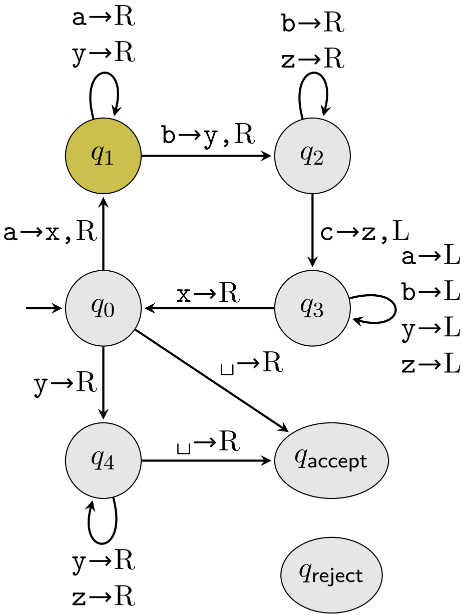

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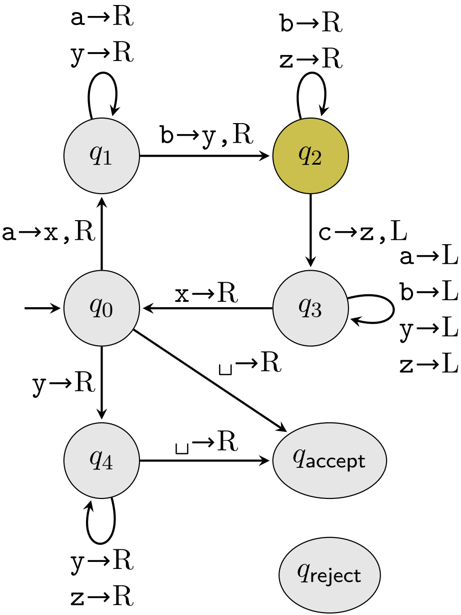
x	a	a	y	b	b	c	c	c			
				↑							



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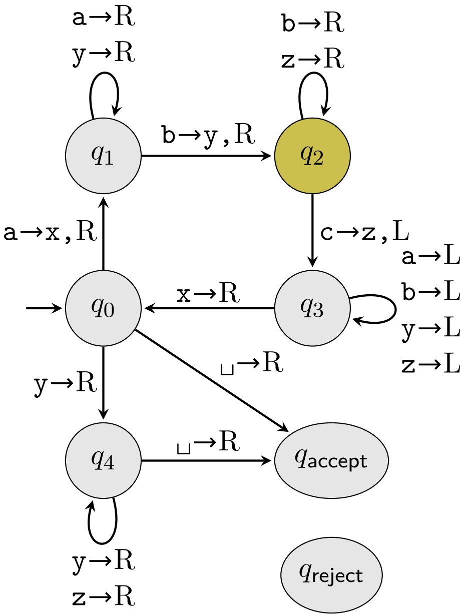

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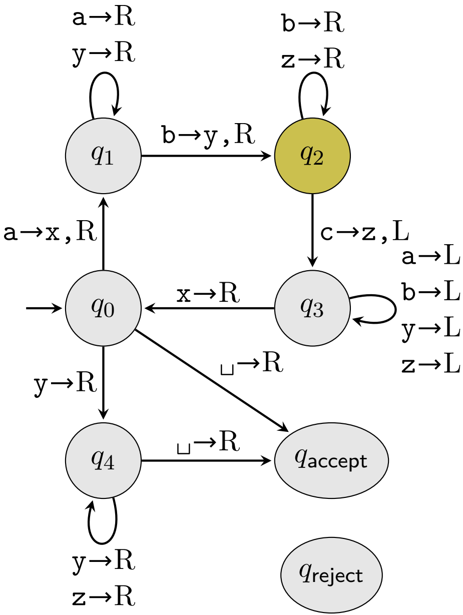

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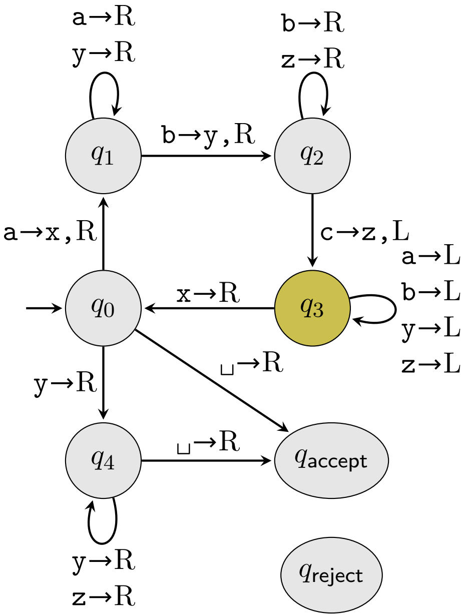

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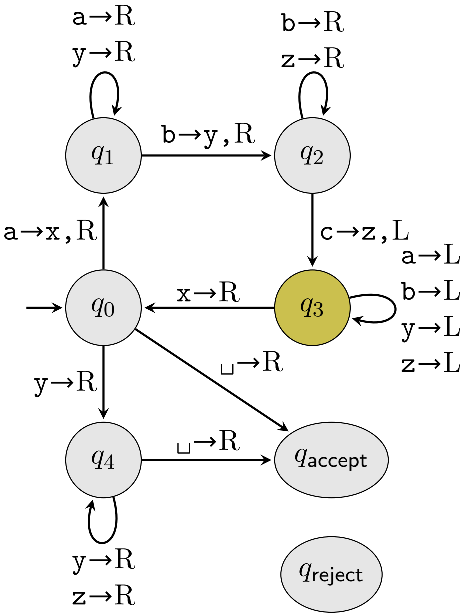

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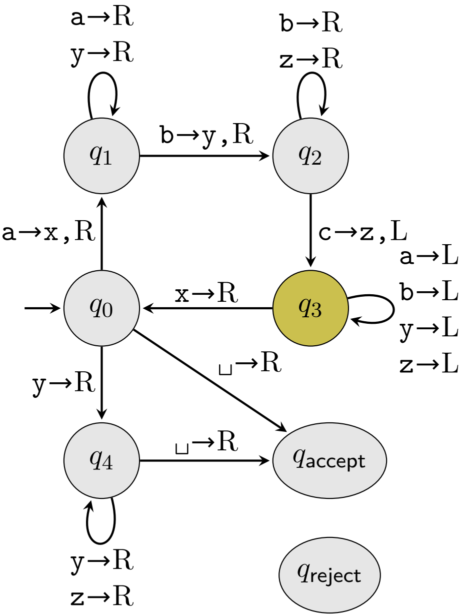

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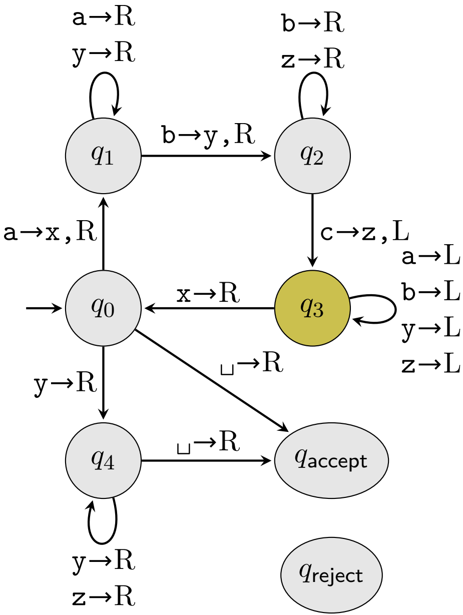

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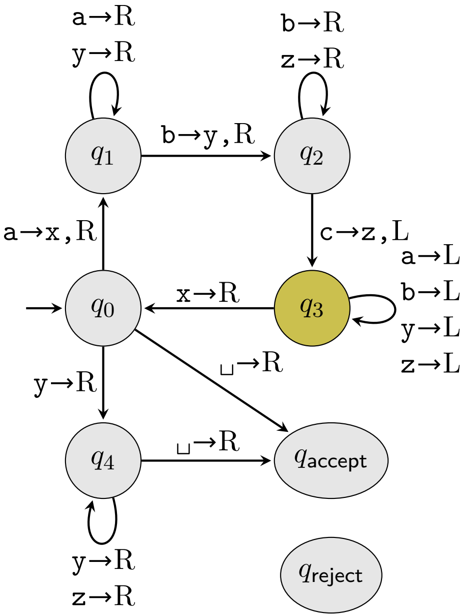

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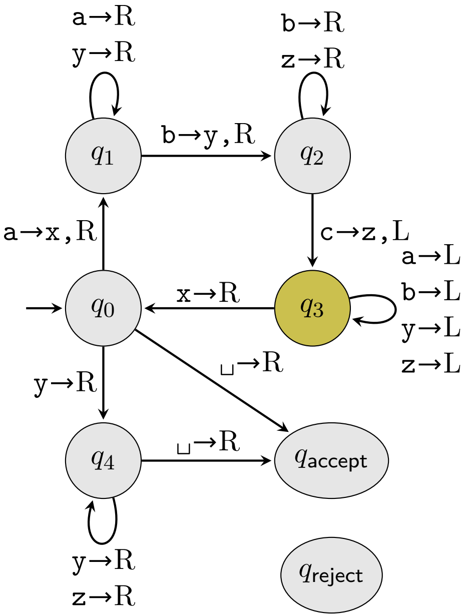

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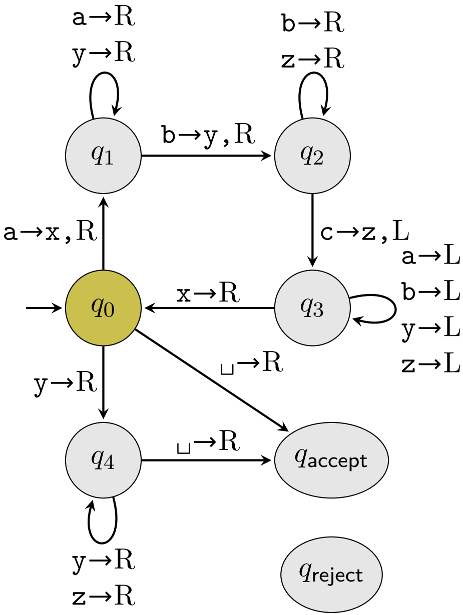

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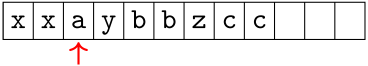
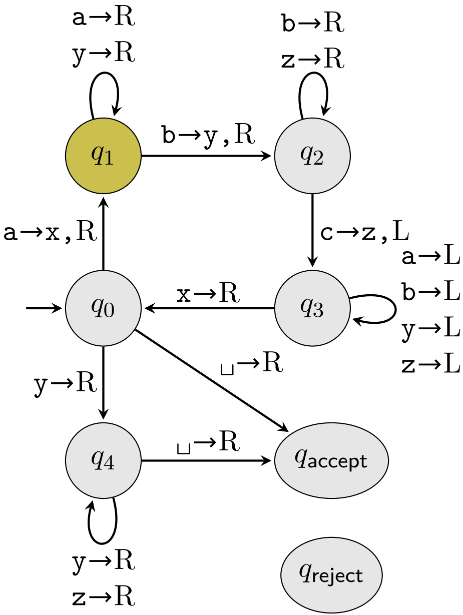


x	x	a	y	b	b	z	c	c			



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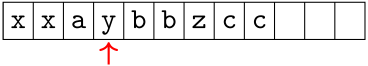

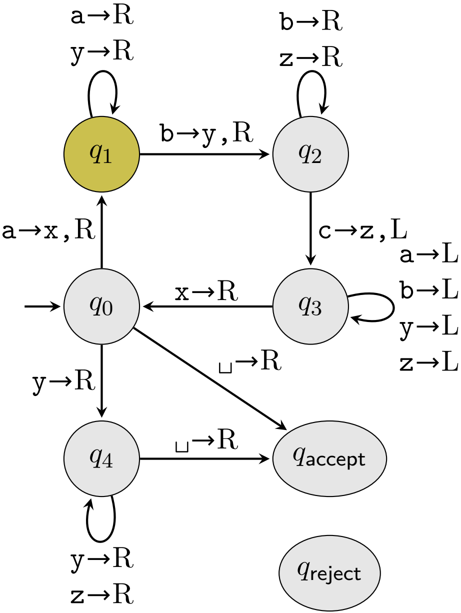

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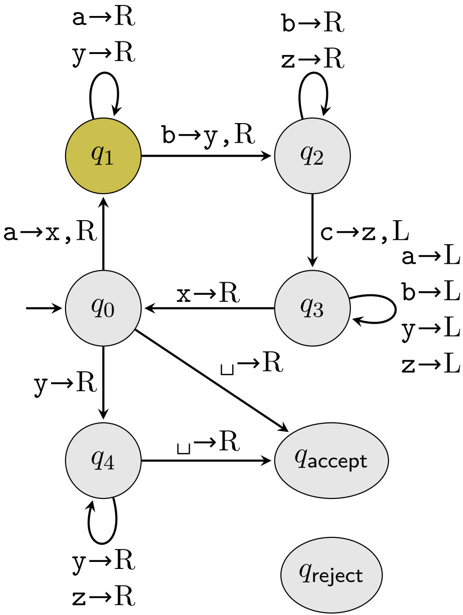

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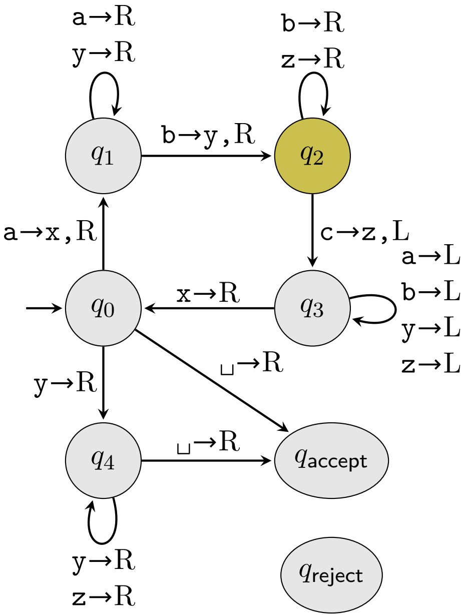

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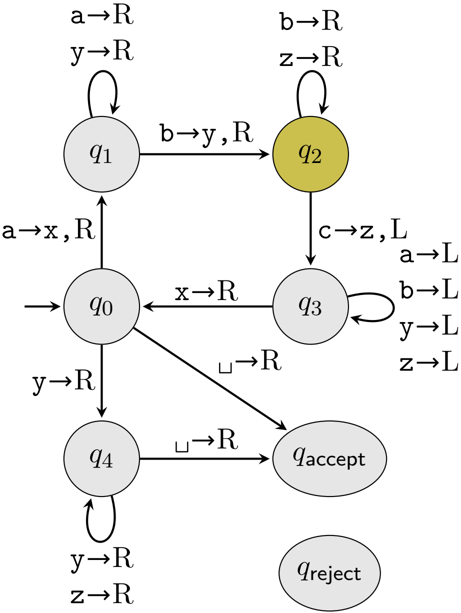

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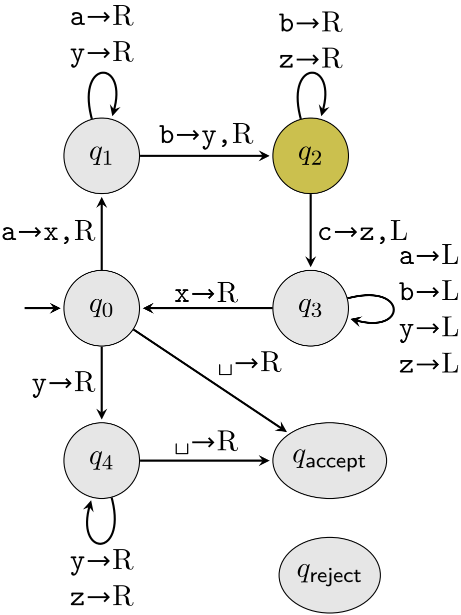

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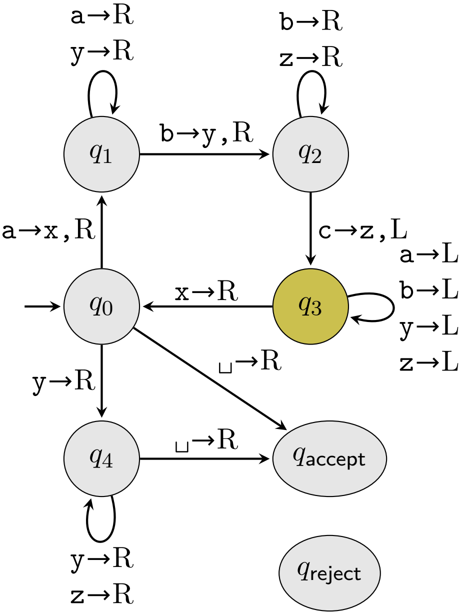
x	x	a	y	y	b	z	z	c			
						↑					

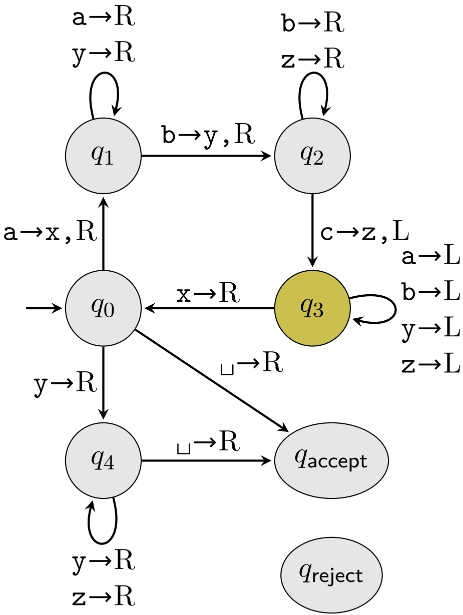


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	a	y	y	b	z	z	c			
					↑						

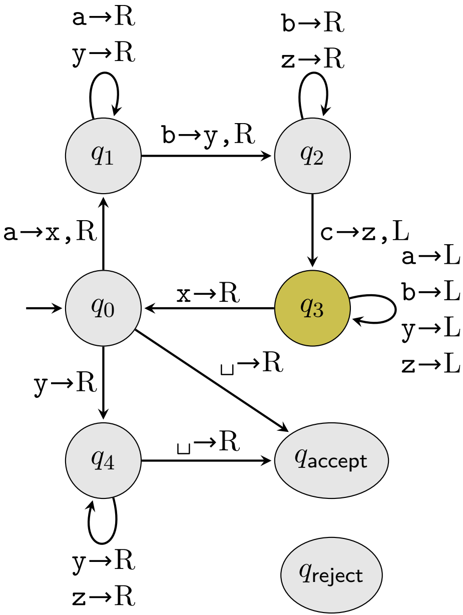


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	a	y	y	b	z	z	c			
				↑							

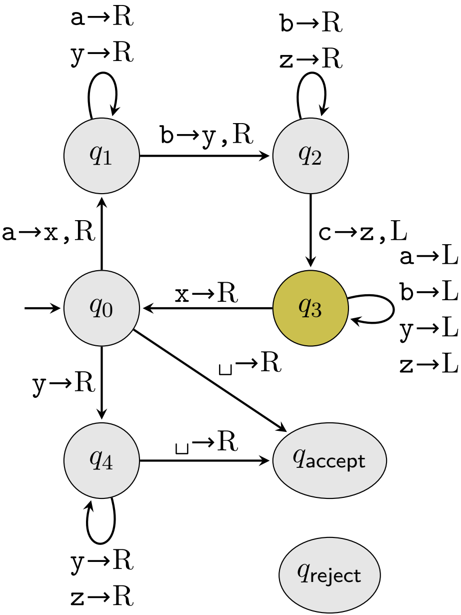


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	a	y	y	b	z	z	c			

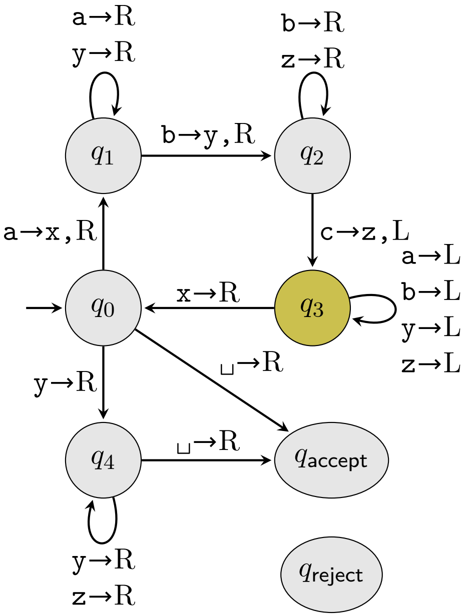


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	a	y	y	b	z	z	c			
		↑									

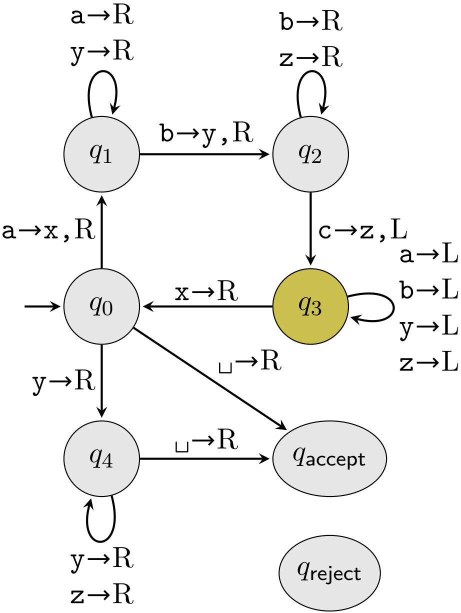


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	a	y	y	b	z	z	c			
	↑										

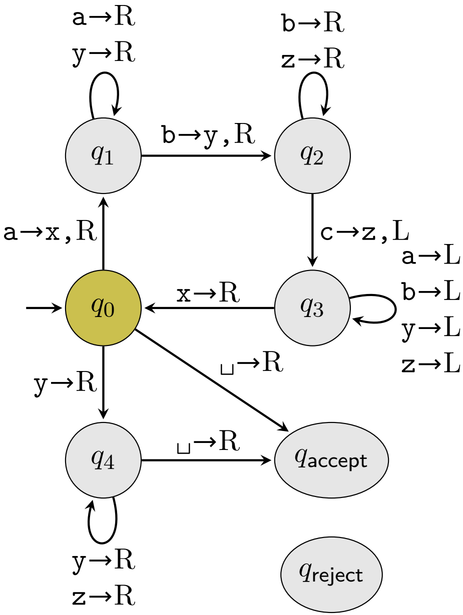


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	a	y	y	b	z	z	c			
		↑									

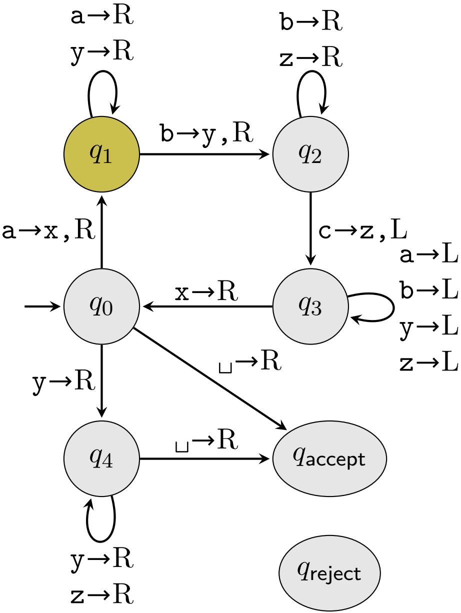


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aabbbccc}$

x	x	x	y	y	b	z	z	c			
			↑								

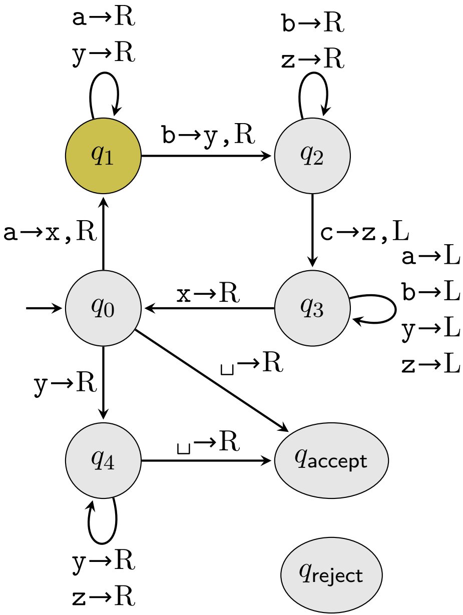


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	x	y	y	b	z	z	c			
				↑							

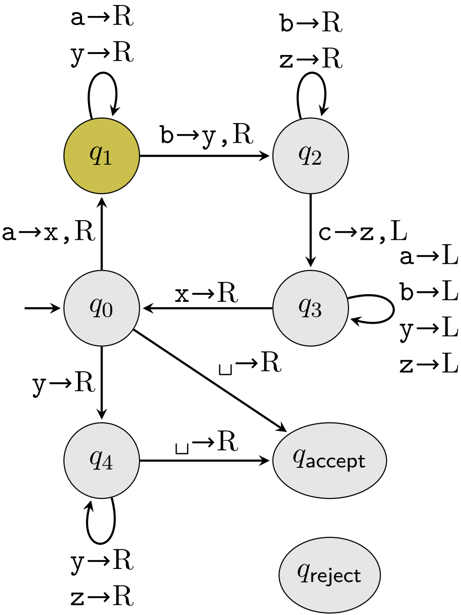


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	x	y	y	b	z	z	c			
					↑						

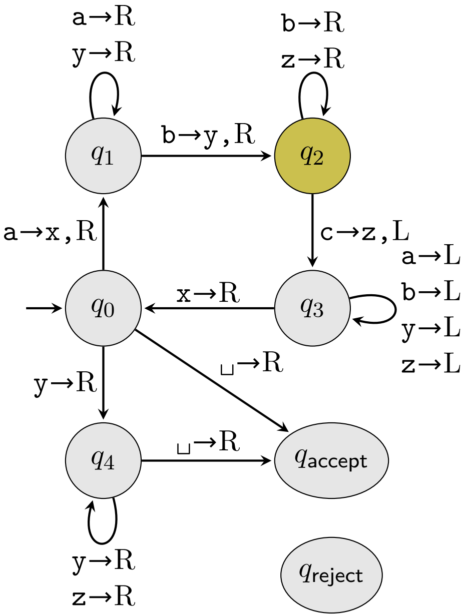


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	x	y	y	y	z	z	c			
						↑					

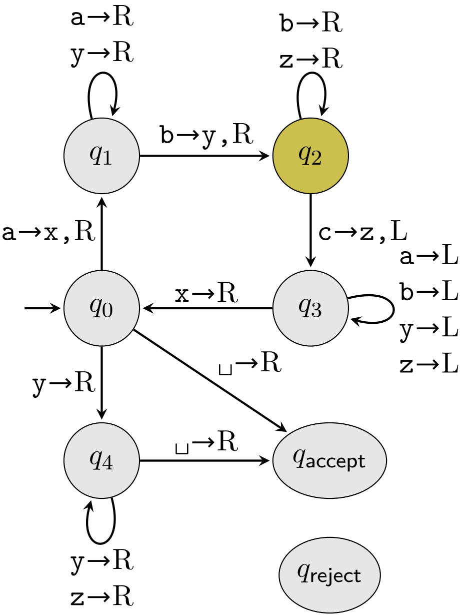


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	x	y	y	y	z	z	c			

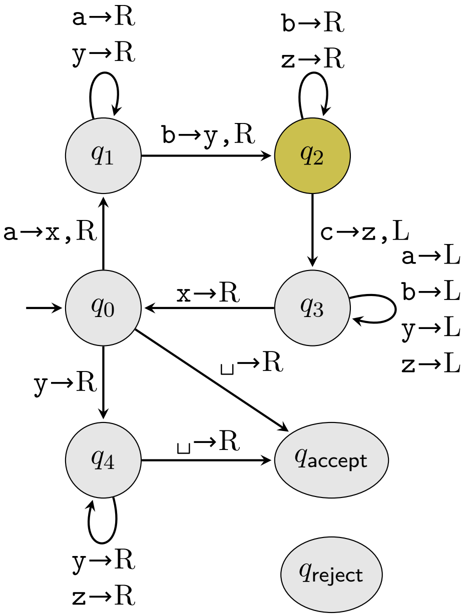


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	x	y	y	y	z	z	c			

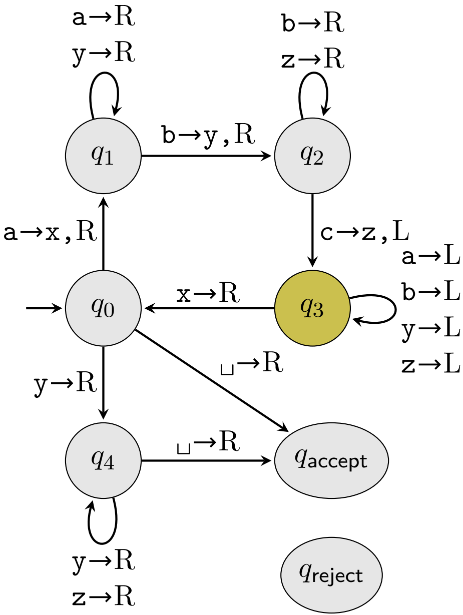


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

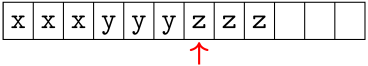
x	x	x	y	y	y	z	z	z			
							↑				



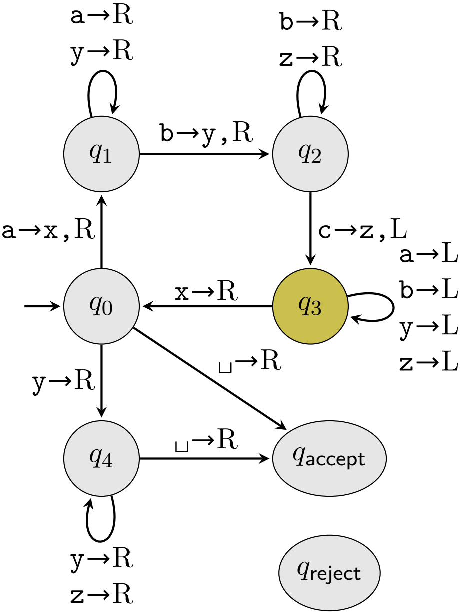
Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$



x	x	x	y	y	y	z	z	z			

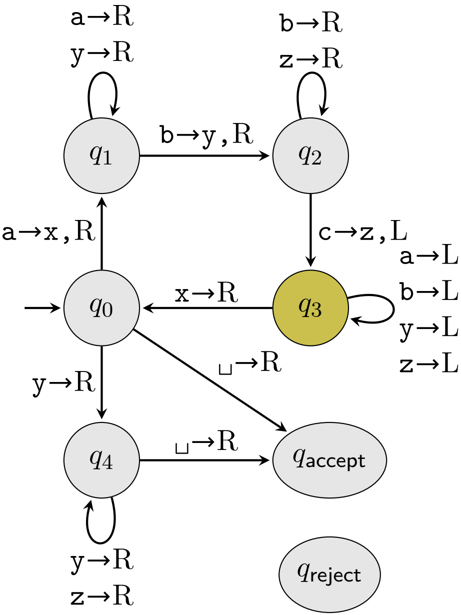


Example

$A = \{\texttt{a}^n\texttt{b}^n\texttt{c}^n \mid n \geq 0\}$

$w = \texttt{aaabbbccc}$

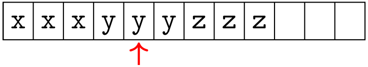
x	x	x	y	y	y	z	z	z			
					↑						



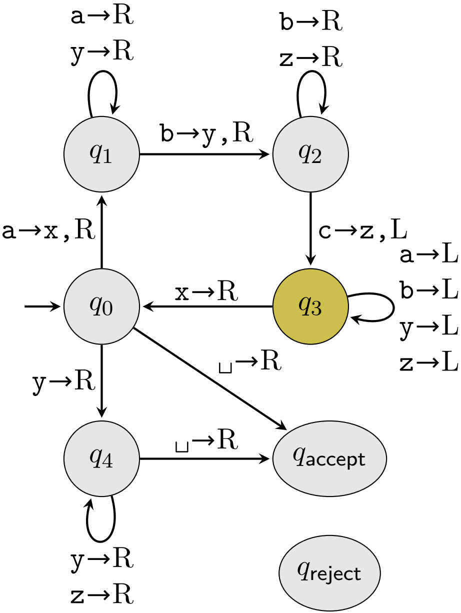
Example

$A = \{\texttt{a}^n\texttt{b}^n\texttt{c}^n \mid n \geq 0\}$

$w = \texttt{aaabbbccc}$



x	x	x	y	y	y	z	z	z			

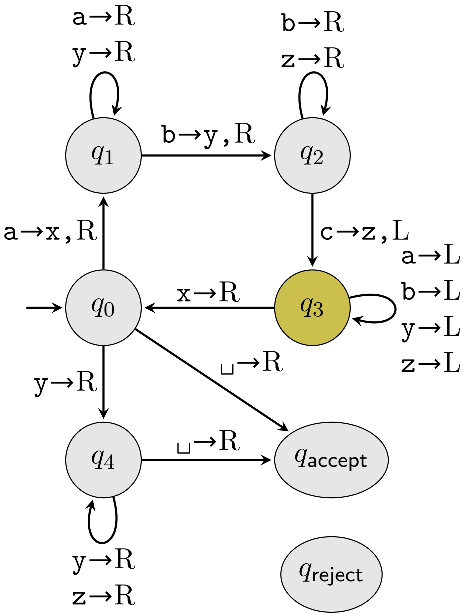


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

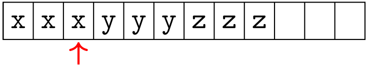
x	x	x	y	y	y	z	z	z			
			↑								



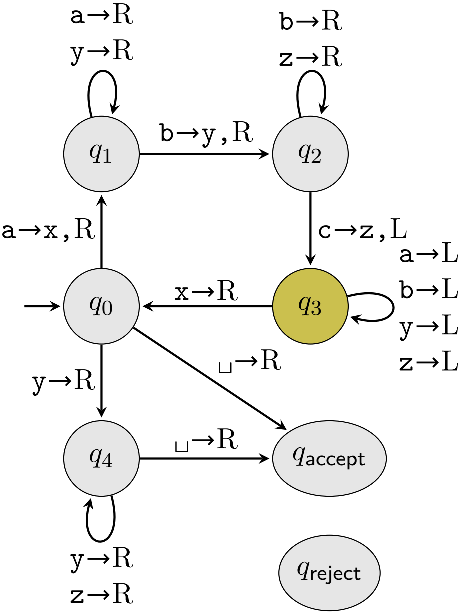
Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$



x	x	x	y	y	y	z	z	z			

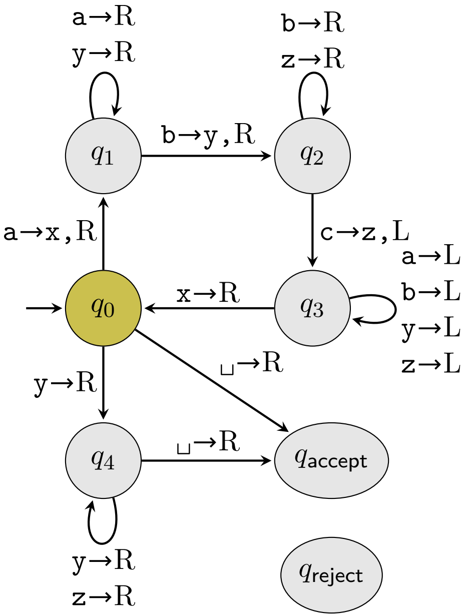


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	x	y	y	y	z	z	z			
			↑								

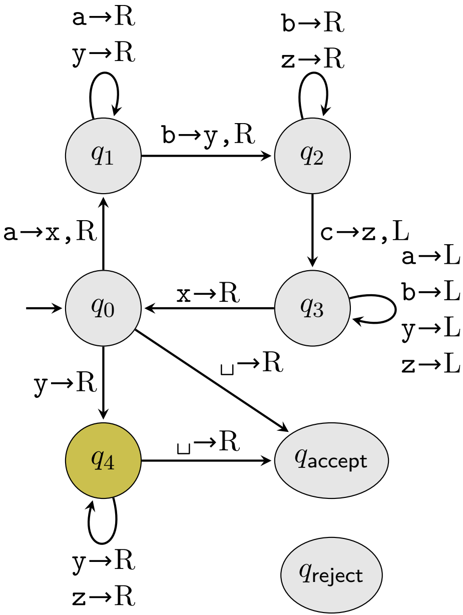


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	x	y	y	y	z	z	z			
				↑							

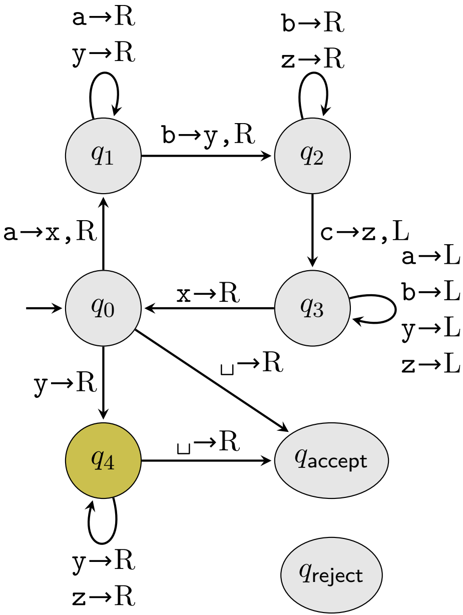


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

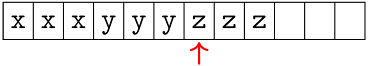
x	x	x	y	y	y	z	z	z			
					↑						



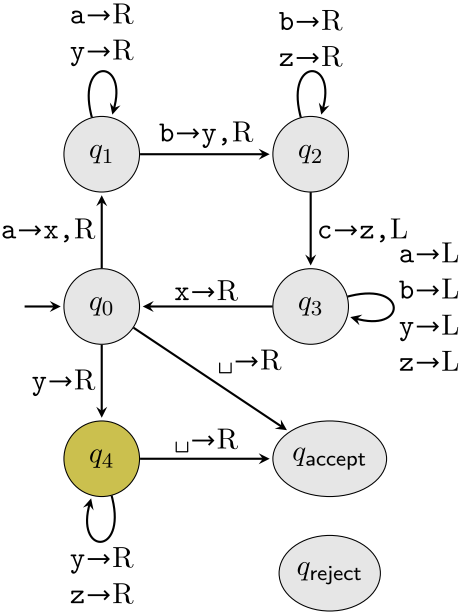
Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$



x	x	x	y	y	y	z	z	z			

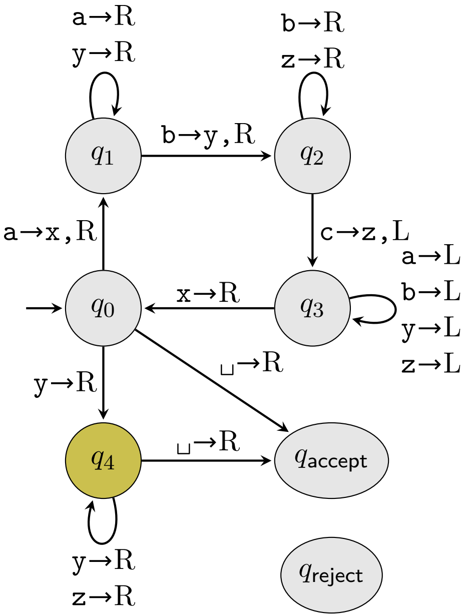


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	x	y	y	y	z	z	z			
							↑				

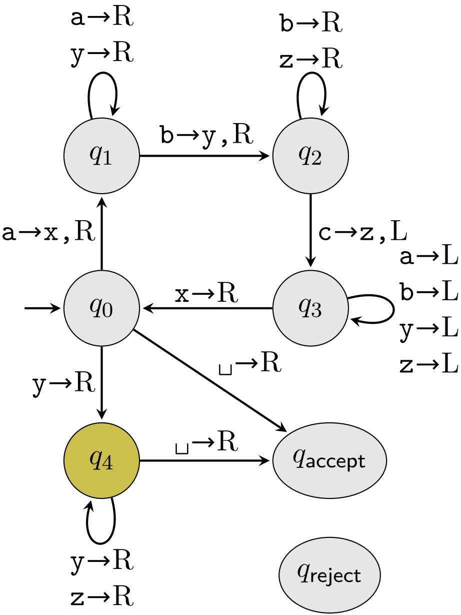


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

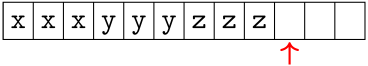
x	x	x	y	y	y	z	z	z			
								↑			



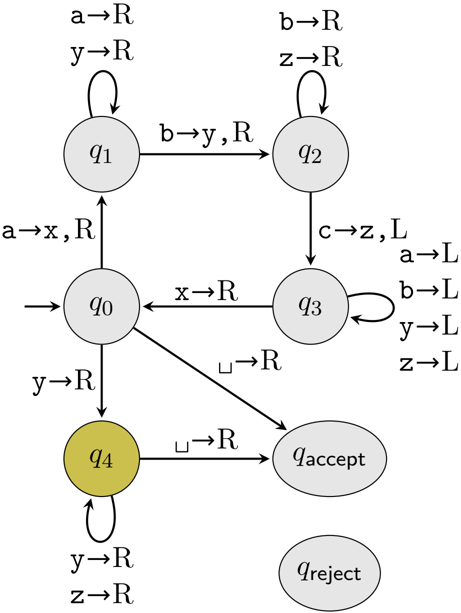
Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$



x	x	x	y	y	y	z	z	z			

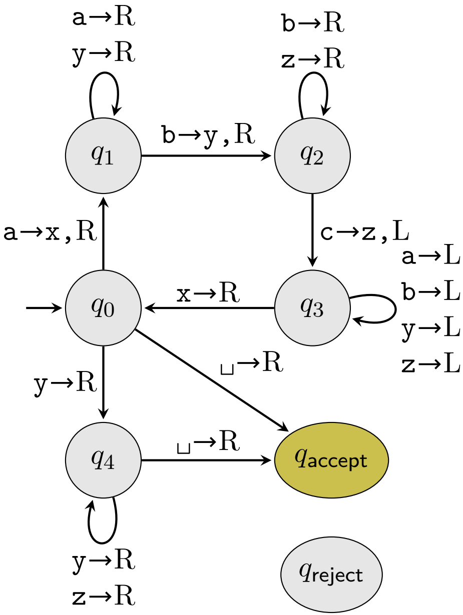


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{aaabbbccc}$

x	x	x	y	y	y	z	z	z			
										↑	

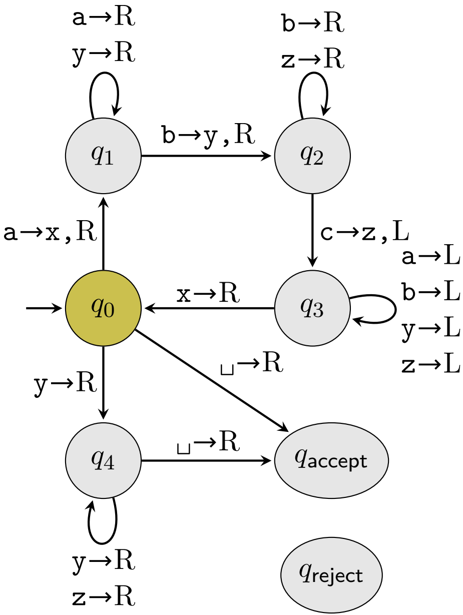


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{abac}$

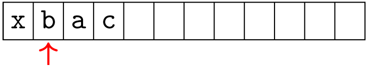
a	b	a	c								
↑											

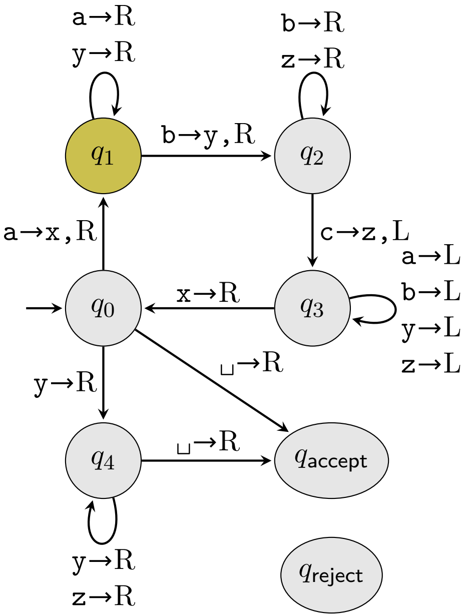


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{abac}$



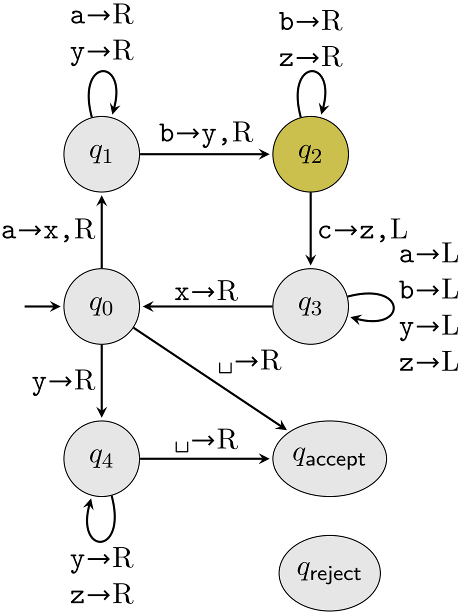


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{abac}$

x	y	a	c								
		↑									

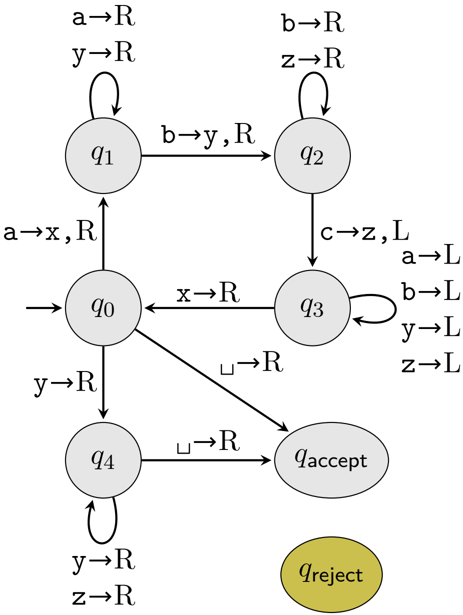


Example

$A = \{\texttt{a}^n\texttt{b}^n\texttt{c}^n \mid n \geq 0\}$

$w = \texttt{abac}$

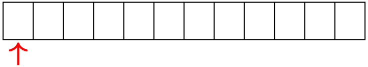
x	y	a	c								
			↑								

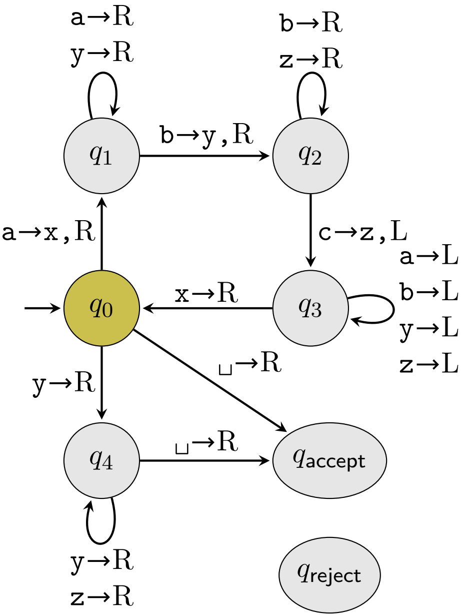


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \varepsilon$

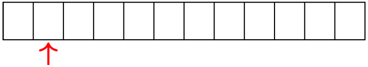


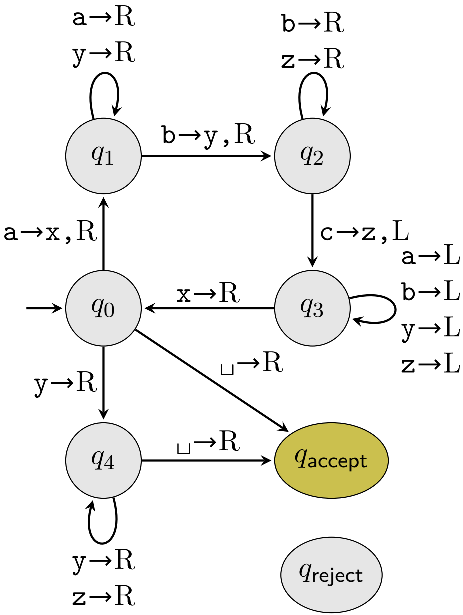


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \varepsilon$



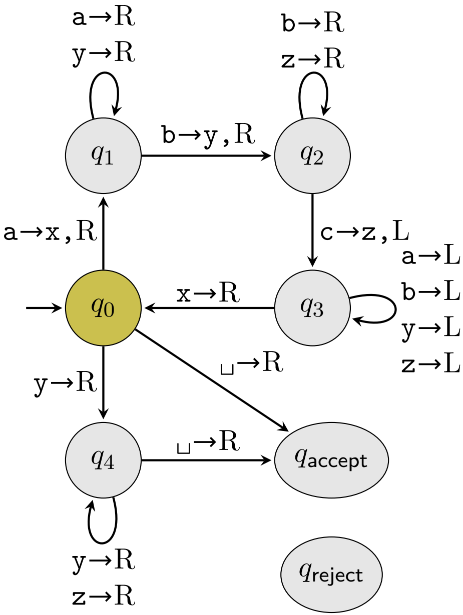


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{abbc}$

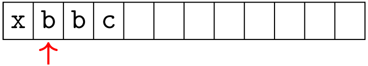
a	b	b	c								
↑											

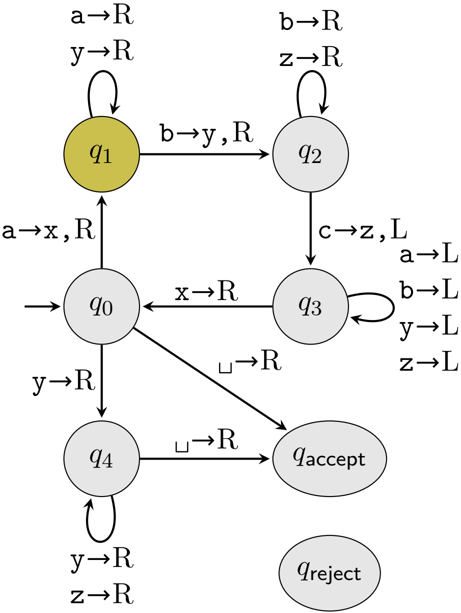


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{abbc}$



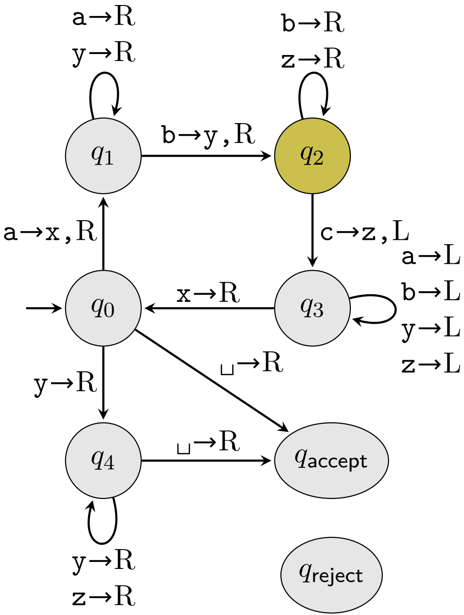


Example

$A = \{\texttt{a}^n\texttt{b}^n\texttt{c}^n \mid n \geq 0\}$

$w = \texttt{abbc}$

x	y	b	c								
		↑									

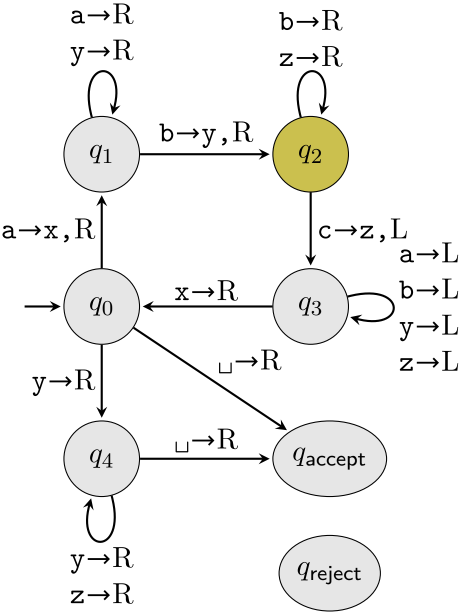


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{abbc}$

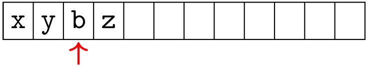
x	y	b	c								
			↑								

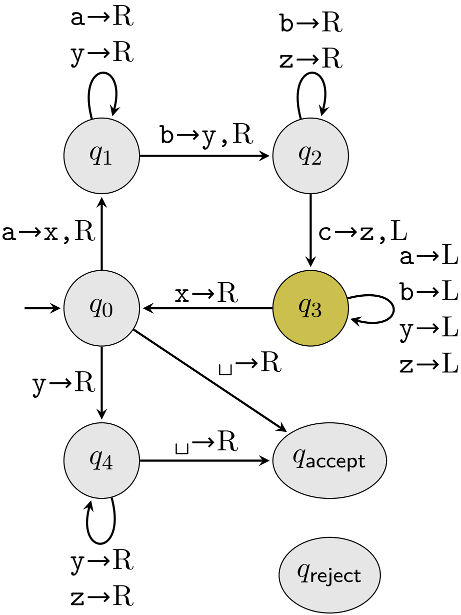


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{abbc}$

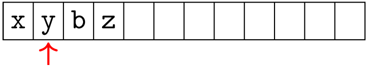


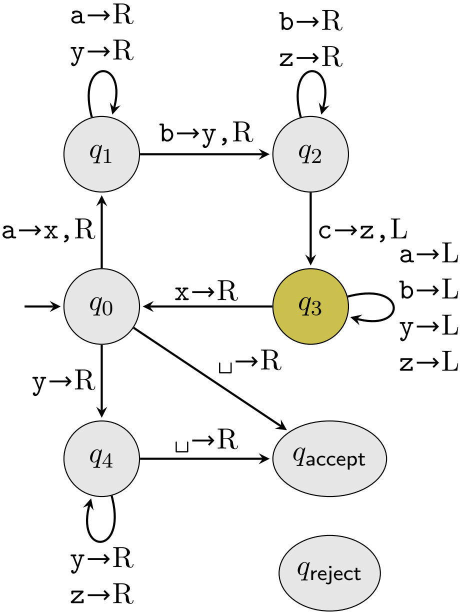


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{abbc}$



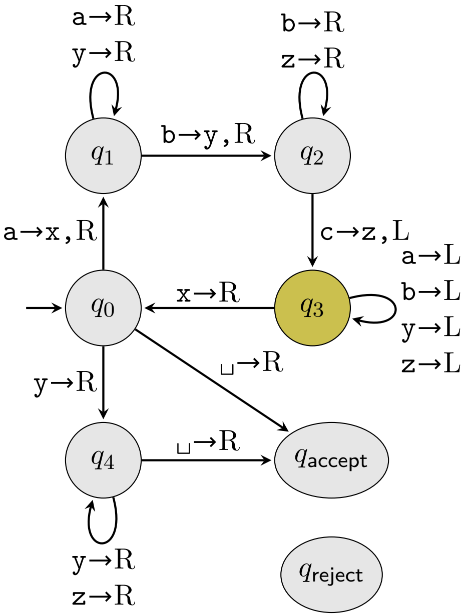


Example

$A = \{\texttt{a}^n\texttt{b}^n\texttt{c}^n \mid n \geq 0\}$

$w = \texttt{abbc}$

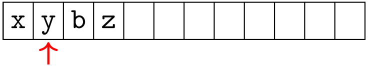
x	y	b	z								
↑											

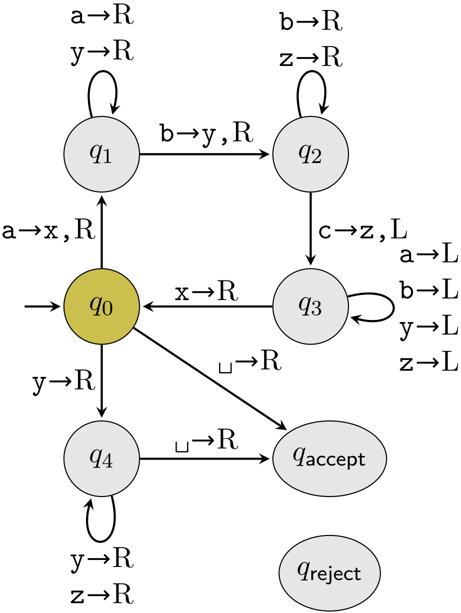


Example

$A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

$w = \mathtt{abbc}$

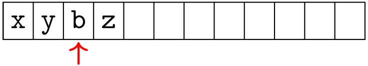


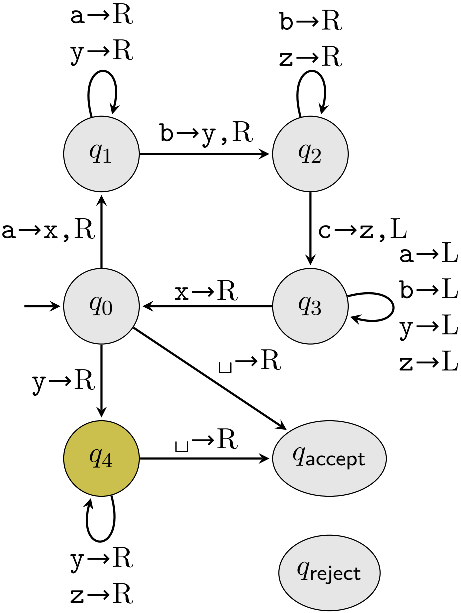


Example

$A = \{\texttt{a}^n\texttt{b}^n\texttt{c}^n \mid n \geq 0\}$

$w = \texttt{abbc}$





Example

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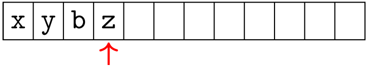
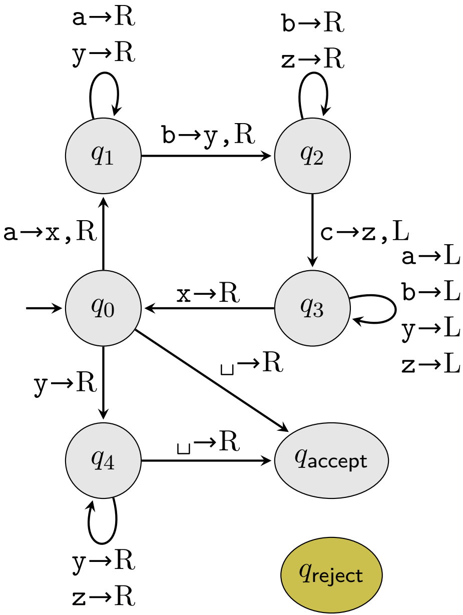


x	y	b	z								



Mathematically

A TM is a 7-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$ where

Q	finite set of states
Σ	finite input alphabet $\sqcup \notin \Sigma$
Γ	finite tape alphabet $\Sigma \subseteq \Gamma$ and $\sqcup \in \Gamma$
δ	transition function
q_0	initial state $q_0 \in Q$
q_{accept}	single accepting state $q_{\text{accept}} \in Q$
q_{reject}	single rejecting state $q_{\text{reject}} \in Q$

TM transition function

$$\delta : (Q \setminus \{q_{\mathsf{accept}}, q_{\mathsf{reject}}\}) \times \Gamma \to Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$$

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For every other state q and tape symbol $s \in \Gamma$ (including $s = \sqcup$),

$$(r, t, D) = \delta(q, s)$$

means: if the cell under the tape head is s, the TM

1. changes to state r;
2. replaces s on the tape with t; and
3. moves the tape head in the direction $D \in \{\mathrm{L}, \mathrm{R}\}$

Turing-recognizable languages

The language of a TM M, denoted $L(M)$ is the set of strings accepted by M
Similarly, M recognizes a language A if $L(M) = A$

[1] Sipser introduced the terminology "Turing-recognizable." Other textbooks and papers call this "recursively enumerable," hence RE. You can think Turing-REcognizable.

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A language A is Turing-recognizable (abbreviated RE[1]) if some TM recognizes it

Example:
$\{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$ is RE because we constructed a TM to recognize it

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Decidable languages

For a given string $w \in \Sigma^*$, if TM M either accepts or rejects the string, we say that M halts on w

Otherwise, we say M loops on w because it must have entered an infinite loop

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Decidable languages

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A TM M is called a decider if M halts on every input

A decider that recognizes a language is said to decide it.

I.e.., "TM M decides A" means "$L(M) = A$ and on every input M halts"

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A language is decidable[2] if some TM decides it

Example:

$\{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$ is decidable because we built a decider that recognizes it

Equivalently, it's decidable because we built a TM that decides it

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Decidability is a property of the language

Note that being decidable is a property of the *language*, not of a TM

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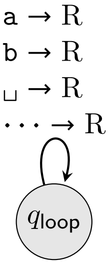
Given any TM M, we can add a new state to M that contains a loop that moves the head right for every $t \in \Gamma$ and replace all transitions to q_{reject} to this looping state



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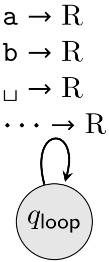

Now, instead of rejecting strings, M will loop.

If we perform this modification to our decider M for $A = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$, we'll get a TM M' such that $L(M') = A$ that doesn't halt on, e.g., $w =$ ba and thus isn't a decider

Nevertheless, A is decidable

Regular languages are decidable

Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$, we can construct a TM $M' = (Q', \Sigma, \Gamma, \delta', q_0, q_{\text{accept}}, q_{\text{reject}})$ where

$$Q' = Q \cup \{q_{\text{accept}}, q_{\text{reject}}\}$$

$$\Gamma = \Sigma \cup \{\sqcup\}$$

$$\delta'(q, t) = \begin{cases} \big(\delta(q,t), t, \mathrm{R}\big) & \text{if } t \in \Sigma \\ (q_{\text{accept}}, \sqcup, \mathrm{R}) & \text{if } t = \sqcup \text{ and } q \in F \\ (q_{\text{reject}}, \sqcup, \mathrm{R}) & \text{if } t = \sqcup \text{ and } q \notin F \end{cases}$$

Regular languages are decidable

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\begin{aligned}
Q' &= Q \cup \{q_{\text{accept}}, q_{\text{reject}}\} \\
\Gamma &= \Sigma \cup \{\sqcup\} \\
\delta'(q, t) &= \begin{cases} \big(\delta(q,t), t, \mathrm{R}\big) & \text{if } t \in \Sigma \\ (q_{\text{accept}}, \sqcup, \mathrm{R}) & \text{if } t = \sqcup \text{ and } q \in F \\ (q_{\text{reject}}, \sqcup, \mathrm{R}) & \text{if } t = \sqcup \text{ and } q \notin F \end{cases}
\end{aligned}
$$

When M' runs on input w, it moves through the input one symbol at a time because it always moves to the right.

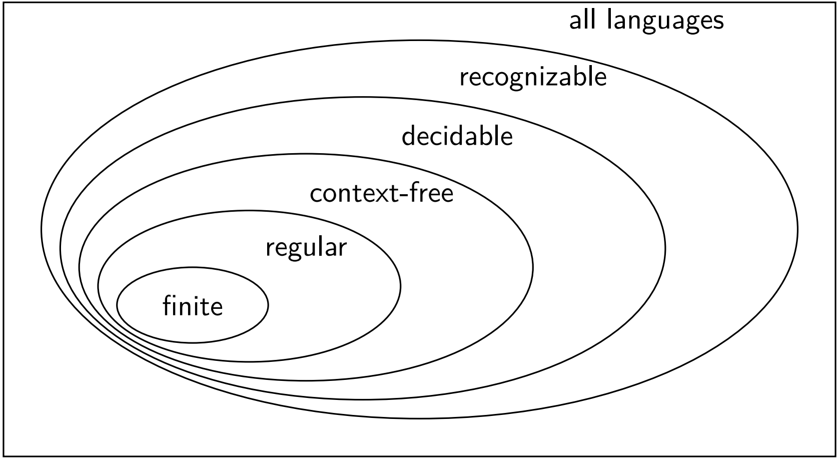
Once M' reaches the end of the input, the cell under the tape head will contain $\sqcup$ and if M' is in a state in F, it will transition to q_{accept}, otherwise it will transition to q_{reject}

Thus $L(M') = L(M)$. Since M' always moves right and halts after the first $\sqcup$, it halts on every input and thus is a decider

Relationship between languages

Later, we'll prove

- every CFL is decidable
- there exist languages that aren't decidable but are Turing-recognizable
- there exist languages that aren't Turing-recognizable



TM configurations

A configuration of a TM is a triple uqv where

$u \in \Gamma^*$ is the tape to the left of the head

$q \in Q$ is the current state

$v \in \Gamma^*$ is the portion of tape under and to the right of the head up to at least the last nonblank cell

The tape itself is uv plus an infinite number of blanks to the right (note that as this is infinite, it's not a string)

Computation of a TM

TM $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$ accepts $w \in \Sigma^*$, if there is a sequence of configurations $c_0, c_1, \ldots, c_n$ where $c_i = u_i n_i v_i$ such that

1. $c_0 = u_0 r_0 v_0$ is the start configuration: $u_0 = \varepsilon$, $r_0 = q_0$, and $v_0 = w$;
2. for $i > 0$, c_i follows from c_{i-1} according to δ; and
3. c_n is an accepting configuration, that is, $r_n = q_{\text{accept}}$

Moving from c_{i-1} to c_i is slightly tricky to write down but conceptually simple

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If $v_{i-1} = \varepsilon$, let $s = \sqcup$ and $v = \varepsilon$, otherwise, let s be the first symbol of v_{i-1} and v be the rest. I.e., $sv = v_{i-1}$
Let $(r, t, D) = \delta(r_{i-1}, s)$

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- If $D = \mathrm{R}$, then $u_i = u_{i-1}t$, $r_i = r$, and $v_i = v$

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- If $D = \mathrm{L}$ and $u_{i-1} = \varepsilon$, then $u_i = \varepsilon$, $r_i = r$, and $v_i = tv$

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- If $D = \mathrm{L}$ and $u_{i-1} = \varepsilon$, then $u_i = \varepsilon$, $r_i = r$, and $v_i = tv$
- If $D = \mathrm{L}$ and $u_{i-1} = ux$ for $x \in \Gamma$ and $u \in \Gamma^*$, then $u_i = u$, $r_i = r$, and $v_i = xtv$

Example configurations

$A = \{\texttt{a}^n\texttt{b}^n\texttt{c}^n \mid n \geq 0\}$

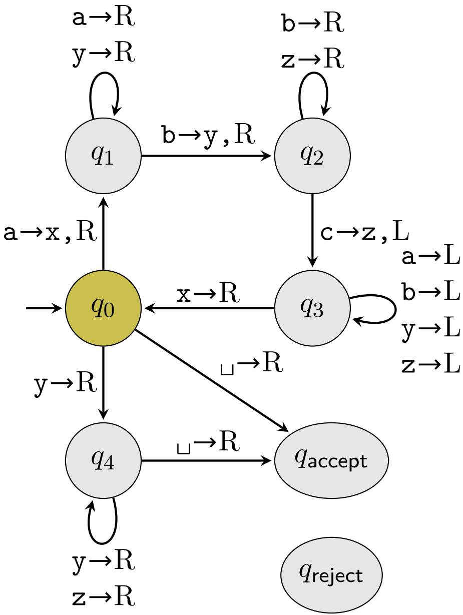
$w = \texttt{abc}$

a	b	c									

↑

Configurations:

0. $q_0\texttt{abc}$



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x	b	c									
	↑										

Configurations:

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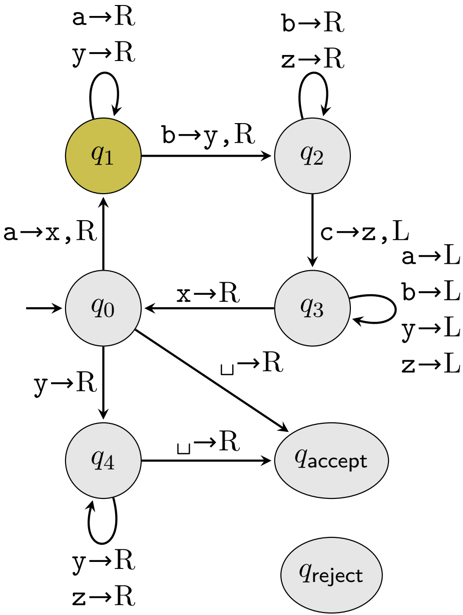

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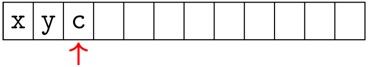

Configurations:

0. $q_0\texttt{abc}$
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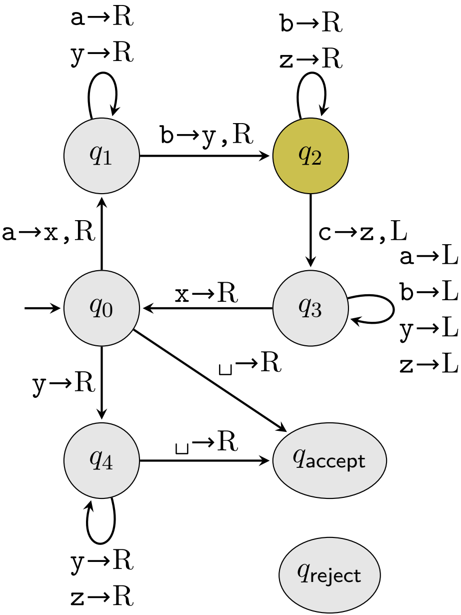

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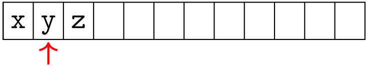

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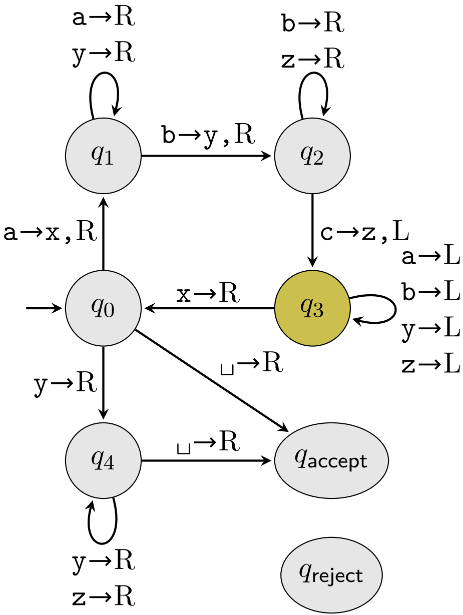
3. $\mathtt{x}q_3\mathtt{yz}$



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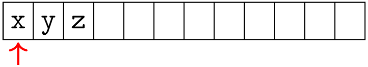

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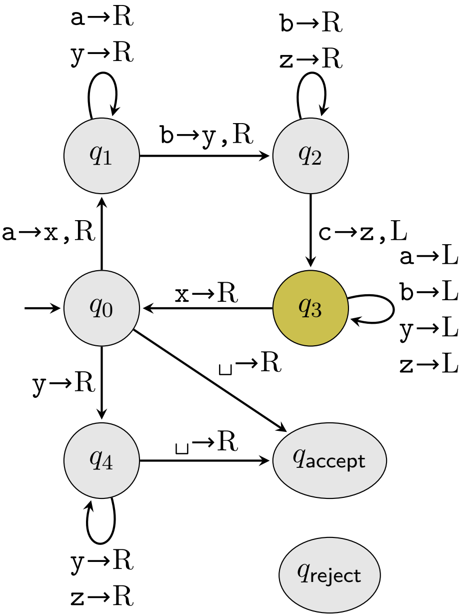
3. $\mathtt{x}q_3\mathtt{yz}$
4. $q_3\mathtt{xyz}$



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x	y	z									
	↑										

Configurations:

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1 $\mathtt{x}q_1\mathtt{bc}$

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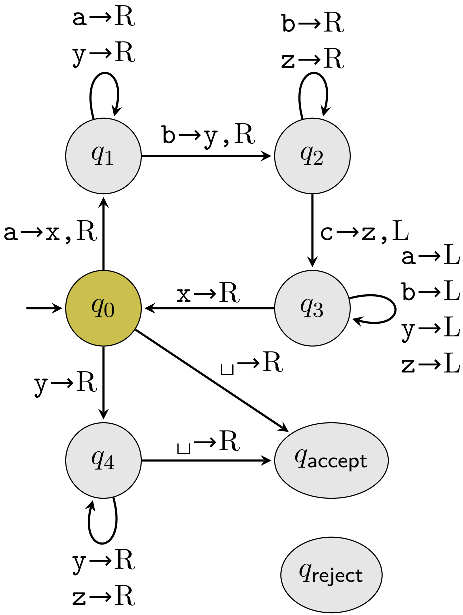
3 $\mathtt{x}q_3\mathtt{yz}$

4 $q_3\mathtt{xyz}$

5 $\mathtt{x}q_0\mathtt{yz}$



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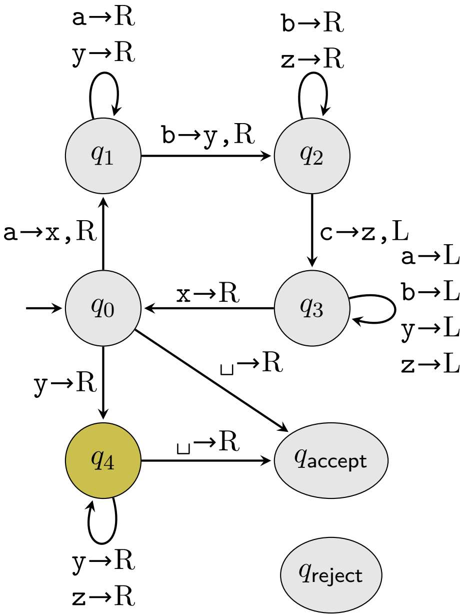
4. $q_3\mathtt{xyz}$
5. $\mathtt{x}q_0\mathtt{yz}$
6. $\mathtt{xy}q_4\mathtt{z}$



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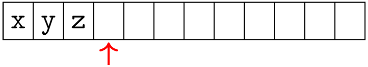

Configurations:

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3. $\mathtt{x}q_3\mathtt{yz}$
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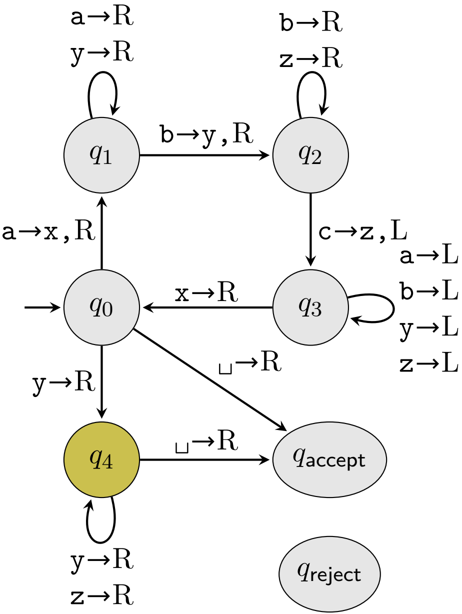
5. $\mathtt{x}q_0\mathtt{yz}$
6. $\mathtt{xy}q_4\mathtt{z}$
7. $\mathtt{xyz}q_4\sqcup$



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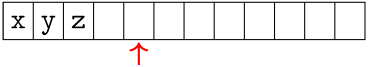

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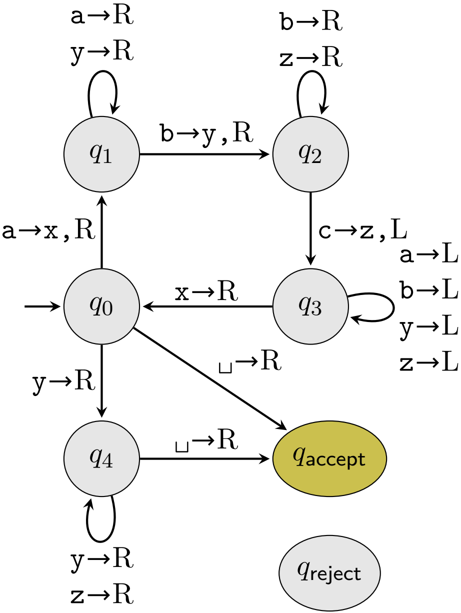
2. $\mathtt{xy}q_2\mathtt{c}$
3. $\mathtt{x}q_3\mathtt{yz}$
4. $q_3\mathtt{xyz}$
5. $\mathtt{x}q_0\mathtt{yz}$
6. $\mathtt{xy}q_4\mathtt{z}$
7. $\mathtt{xyz}q_4\sqcup$
8. $\mathtt{xyz}\sqcup q_{\text{accept}}\sqcup$



Multiplication

$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$

High-level description of a TM to decide B:

- Cross an a off (i.e., replace it with something) and then for each b, cross off a c by moving back and forth between the bs and cs, crossing each off in turn (by replacing b with y and c with z)

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- Once all the bs (and their corresponding cs) have been crossed off, restore all of the b
- Repeat steps 1 and 2 until all of the as have been crossed off
- If any cs remain or if at any point the input was not in the correct order, reject. Otherwise accept.

Multiplication implementation-level description

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Implementation-level description of a TM to decide B:

1. As always, the TM starts with the input on the tape with the head pointing to the first position

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Implementation-level description of a TM to decide B:

1. As always, the TM starts with the input on the tape with the head pointing to the first position
2. If the cell under the head is not a, reject, otherwise replace a with x and scan right past all as

Multiplication implementation-level description

$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$

Implementation-level description of a TM to decide B:

1. As always, the TM starts with the input on the tape with the head pointing to the first position
2. If the cell under the head is not a, reject, otherwise replace a with x and scan right past all as
3. If the cell under the head is not b, reject, otherwise replace b with y and scan right past all bs and zs (which will be the crossed off cs)

Multiplication implementation-level description

$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$

Implementation-level description of a TM to decide B:

1. As always, the TM starts with the input on the tape with the head pointing to the first position
2. If the cell under the head is not a, reject, otherwise replace a with x and scan right past all as
3. If the cell under the head is not b, reject, otherwise replace b with y and scan right past all bs and zs (which will be the crossed off cs)
4. If the cell under the head is not c, reject, otherwise replace c with z and scan left until there is a y (the most recently crossed off b)

Multiplication implementation-level description

$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$

Implementation-level description of a TM to decide B:

1. As always, the TM starts with the input on the tape with the head pointing to the first position
2. If the cell under the head is not a, reject, otherwise replace a with x and scan right past all as
3. If the cell under the head is not b, reject, otherwise replace b with y and scan right past all bs and zs (which will be the crossed off cs)
4. If the cell under the head is not c, reject, otherwise replace c with z and scan left until there is a y (the most recently crossed off b)
5. If the cell under the head is b, goto step 3

Multiplication implementation-level description

$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$

Implementation-level description of a TM to decide B:

1. As always, the TM starts with the input on the tape with the head pointing to the first position
2. If the cell under the head is not a, reject, otherwise replace a with x and scan right past all as
3. If the cell under the head is not b, reject, otherwise replace b with y and scan right past all bs and zs (which will be the crossed off cs)
4. If the cell under the head is not c, reject, otherwise replace c with z and scan left until there is a y (the most recently crossed off b)
5. If the cell under the head is b, goto step 3
6. Otherwise, the cell is a z so scan left, replacing each y with b

Multiplication implementation-level description

$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$

Implementation-level description of a TM to decide B:

1. As always, the TM starts with the input on the tape with the head pointing to the first position
2. If the cell under the head is not a, reject, otherwise replace a with x and scan right past all as
3. If the cell under the head is not b, reject, otherwise replace b with y and scan right past all bs and zs (which will be the crossed off cs)
4. If the cell under the head is not c, reject, otherwise replace c with z and scan left until there is a y (the most recently crossed off b)
5. If the cell under the head is b, goto step 3
6. Otherwise, the cell is a z so scan left, replacing each y with b
7. If the cell under the head is a, scan left to the first a (it's just right of the last x) and return to step 1

Multiplication implementation-level description

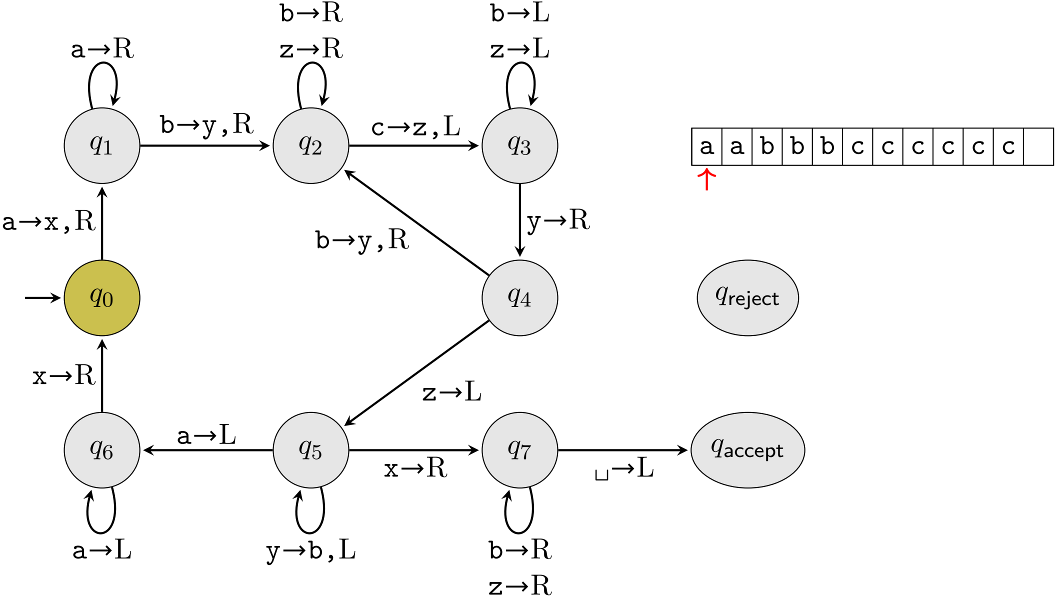
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$

Implementation-level description of a TM to decide B:

1. As always, the TM starts with the input on the tape with the head pointing to the first position
2. If the cell under the head is not a, reject, otherwise replace a with x and scan right past all as
3. If the cell under the head is not b, reject, otherwise replace b with y and scan right past all bs and zs (which will be the crossed off cs)
4. If the cell under the head is not c, reject, otherwise replace c with z and scan left until there is a y (the most recently crossed off b)
5. If the cell under the head is b, goto step 3
6. Otherwise, the cell is a z so scan left, replacing each y with b
7. If the cell under the head is a, scan left to the first a (it's just right of the last x) and return to step 1
8. Otherwise, move right (past the x) and then scan left past all of the bs and zs. If the cell under the head is blank (because all cs have been replaced with zs), accept, otherwise reject

Multiplication TM

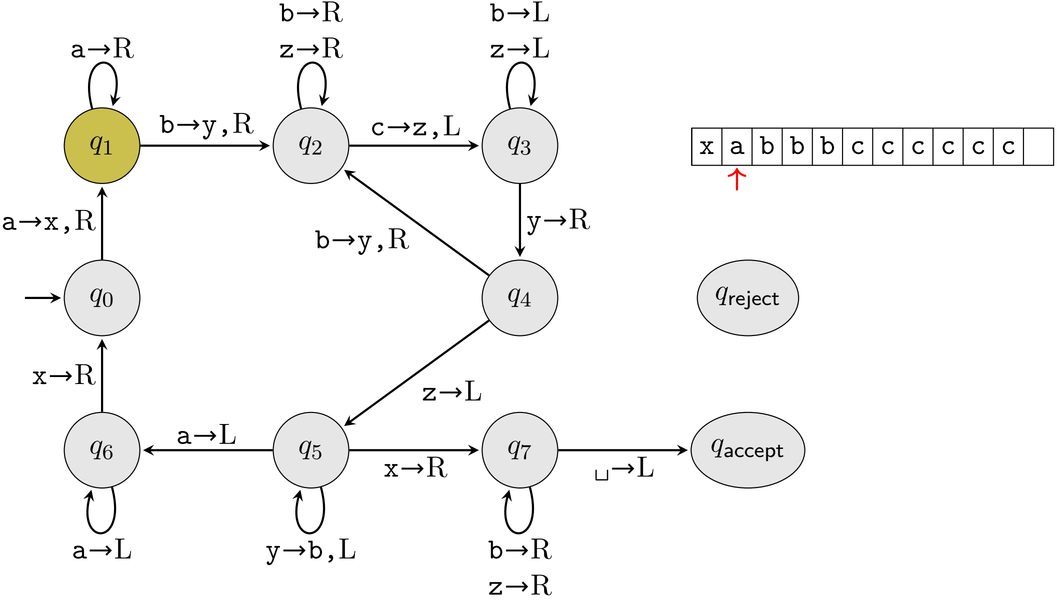
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



a	a	b	b	b	c	c	c	c	c	c	
↑											

Multiplication TM

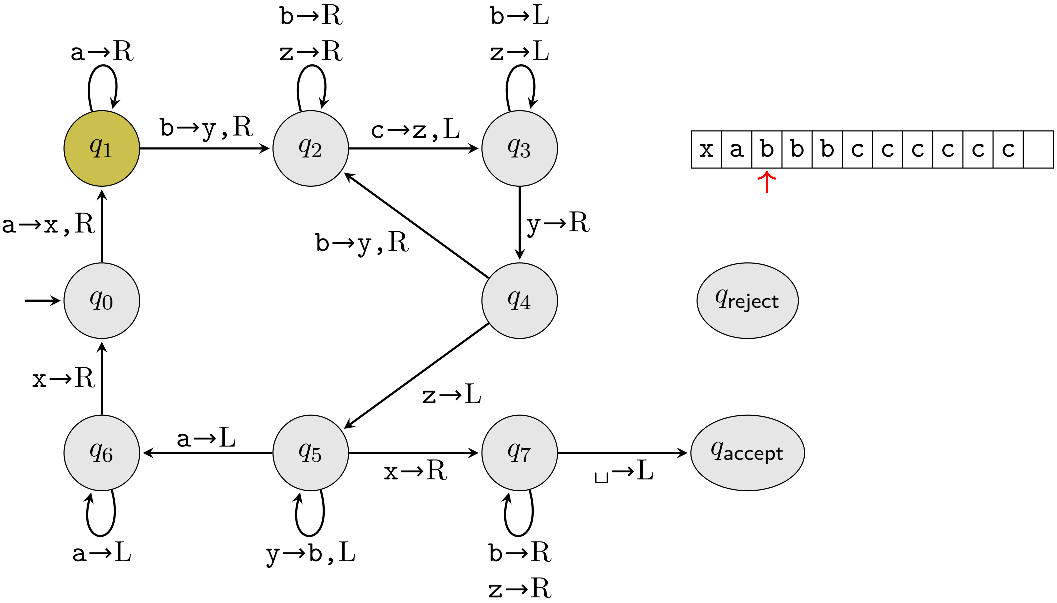
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \mathtt{aabbbcccccc}$



x	a	b	b	b	c	c	c	c	c	c	

Multiplication TM

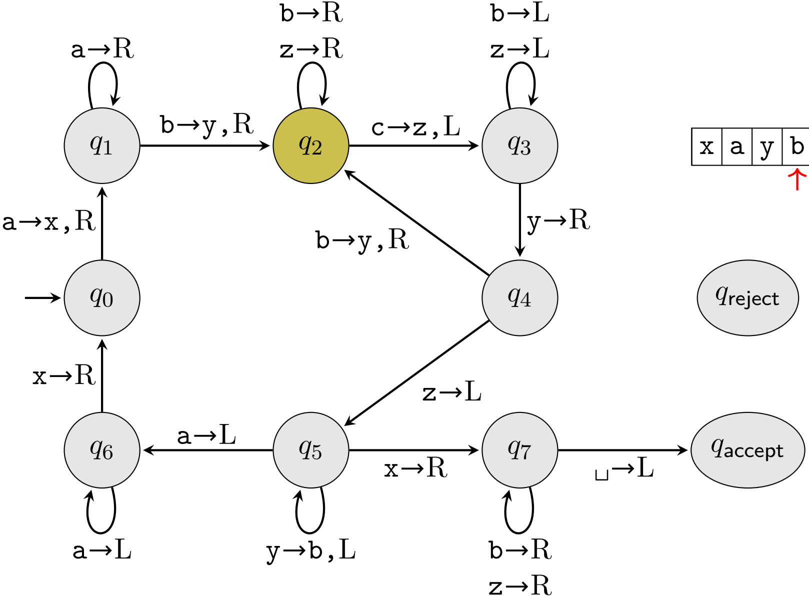
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	b	b	b	c	c	c	c	c	c	
		↑									

Multiplication TM

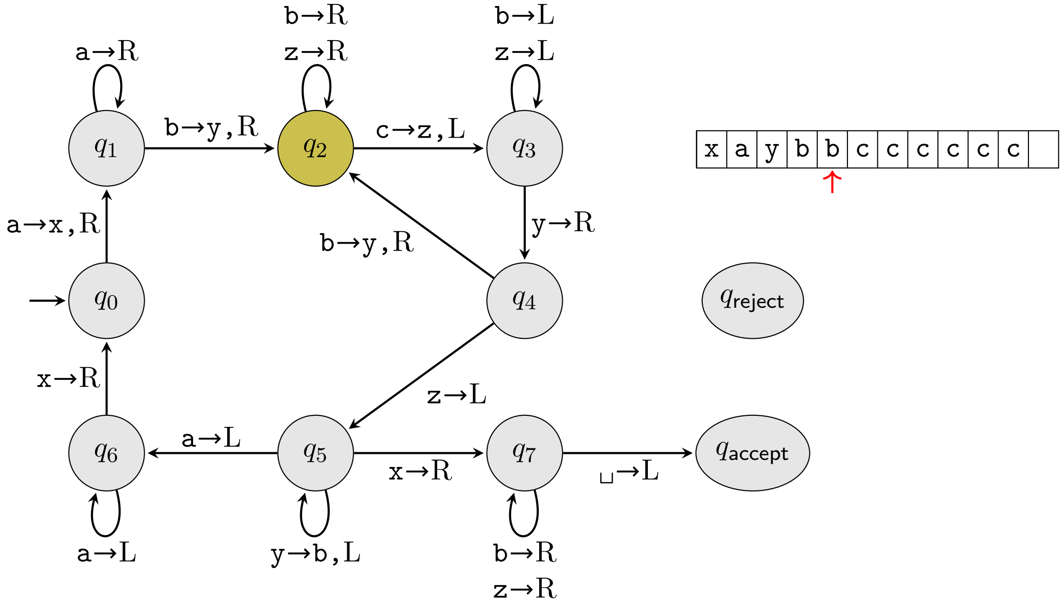
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	y	b	b	c	c	c	c	c	c	
			↑								

Multiplication TM

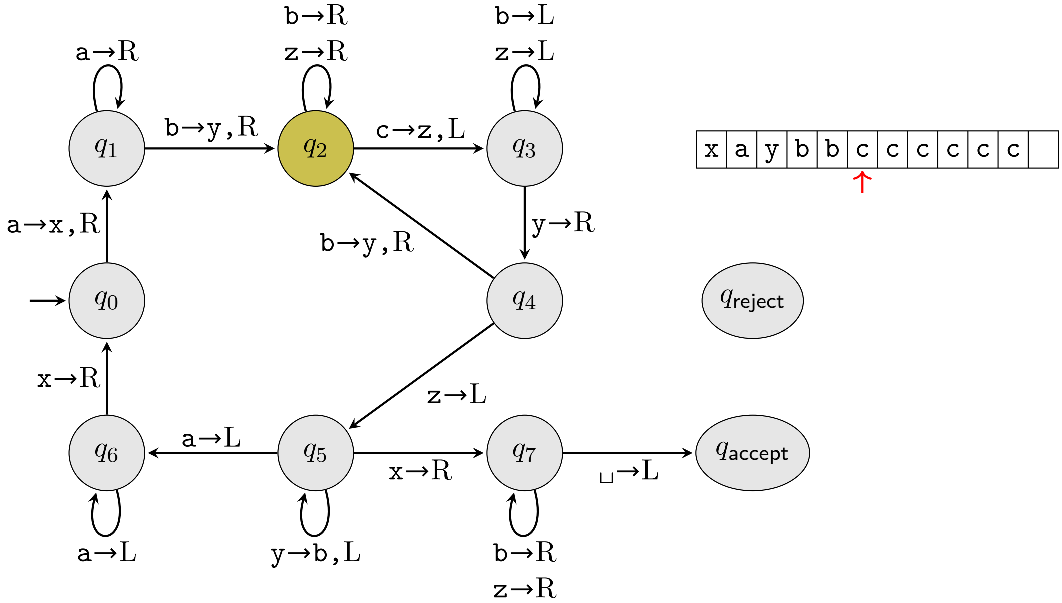
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \texttt{aabbbcccccc}$



x	a	y	b	b	c	c	c	c	c	c	

Multiplication TM

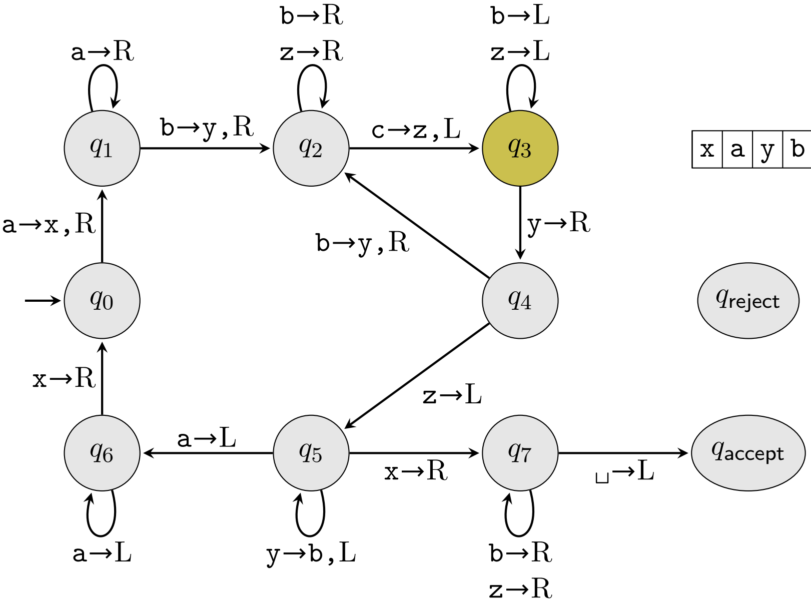
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ w = aabbbcccccc



x	a	y	b	b	c	c	c	c	c	c	

Multiplication TM

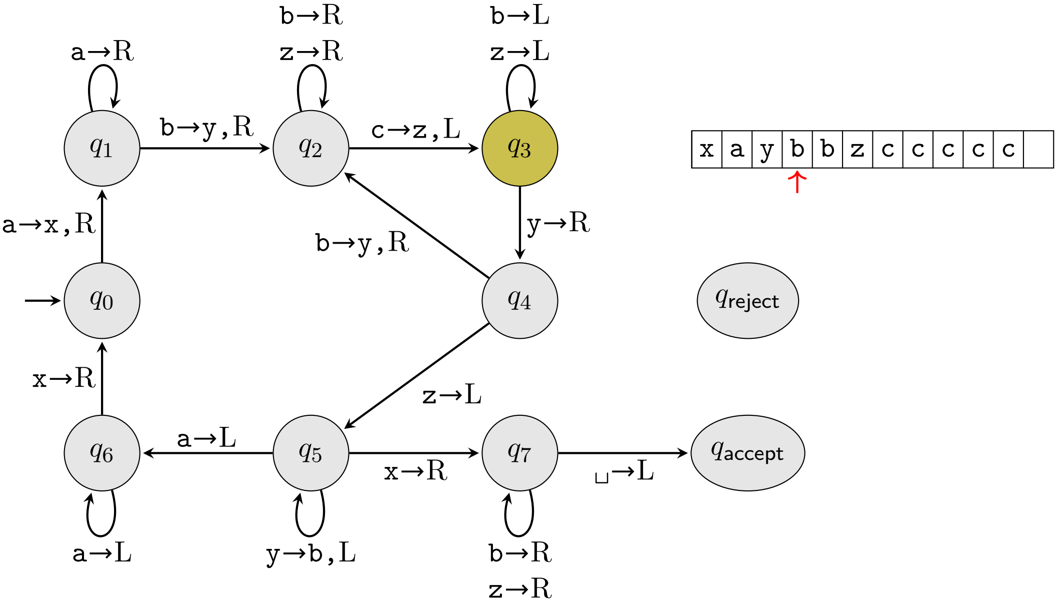
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \mathtt{aabbbcccccc}$



x	a	y	b	b	z	c	c	c	c	c	
				↑							

Multiplication TM

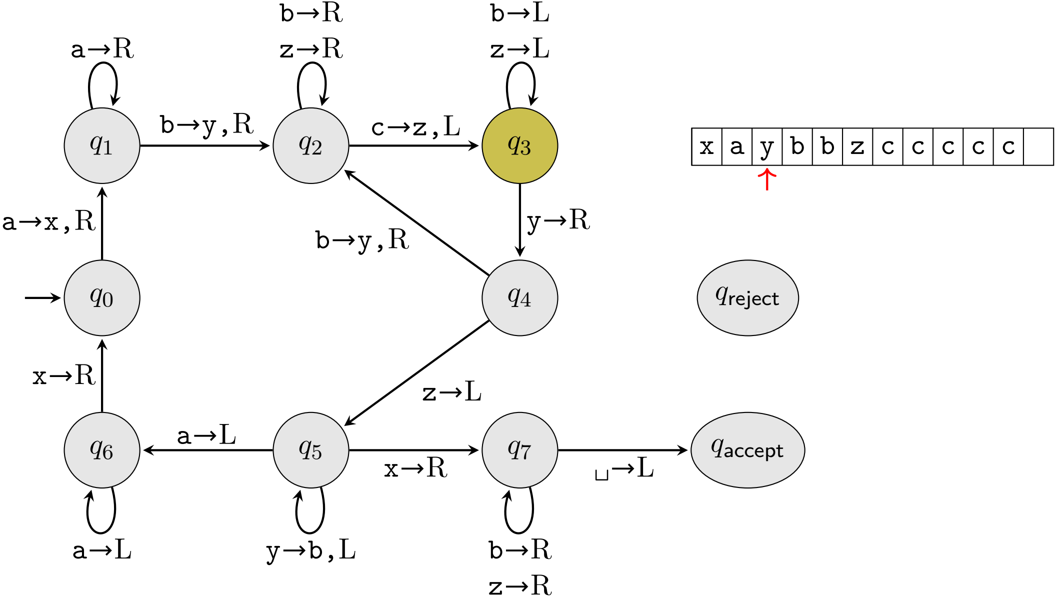
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \mathtt{aabbbcccccc}$



x	a	y	b	b	z	c	c	c	c	c	

Multiplication TM

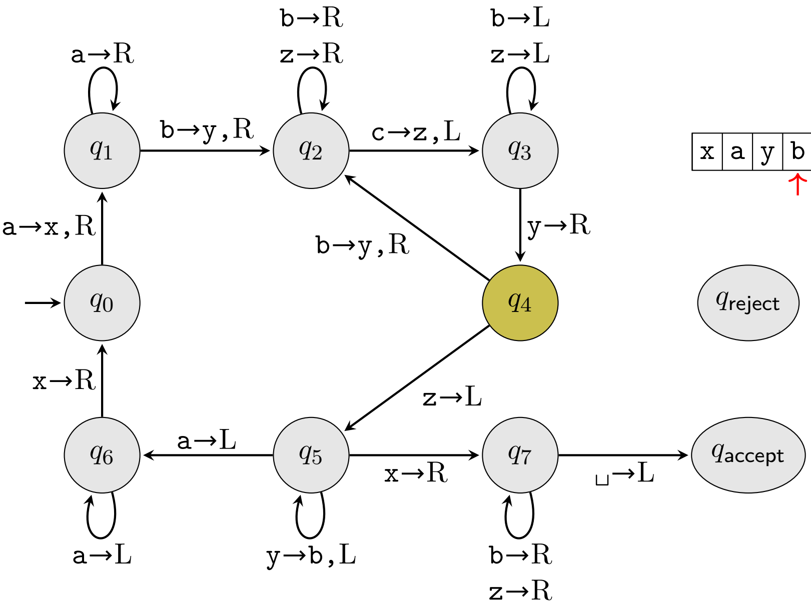
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	y	b	b	z	c	c	c	c	c	

Multiplication TM

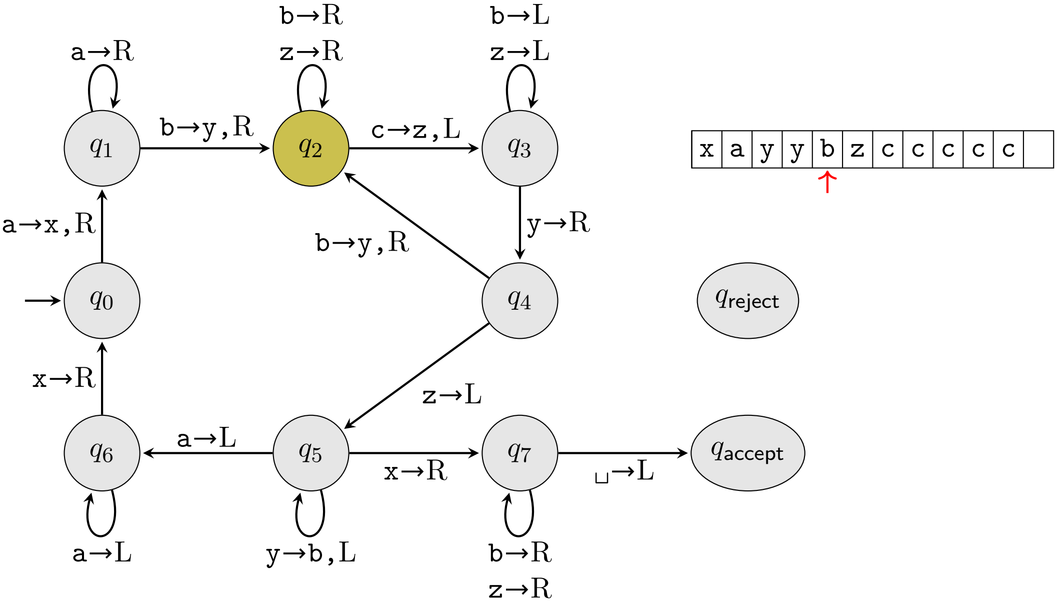
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \texttt{aabbbcccccc}$



x	a	y	b	b	z	c	c	c	c	c	
			↑								

Multiplication TM

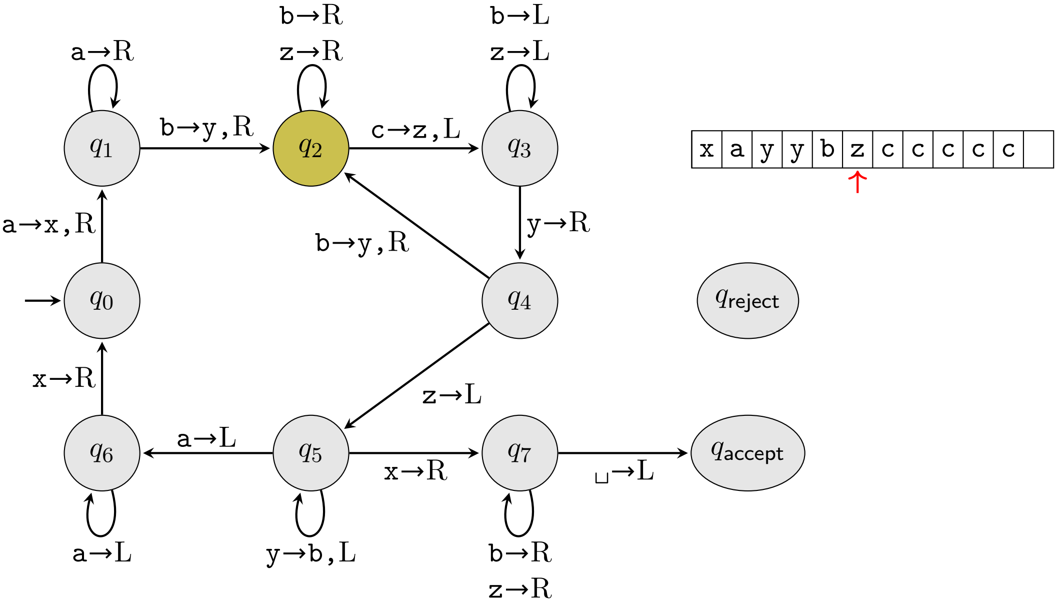
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	y	y	b	z	c	c	c	c	c	

Multiplication TM

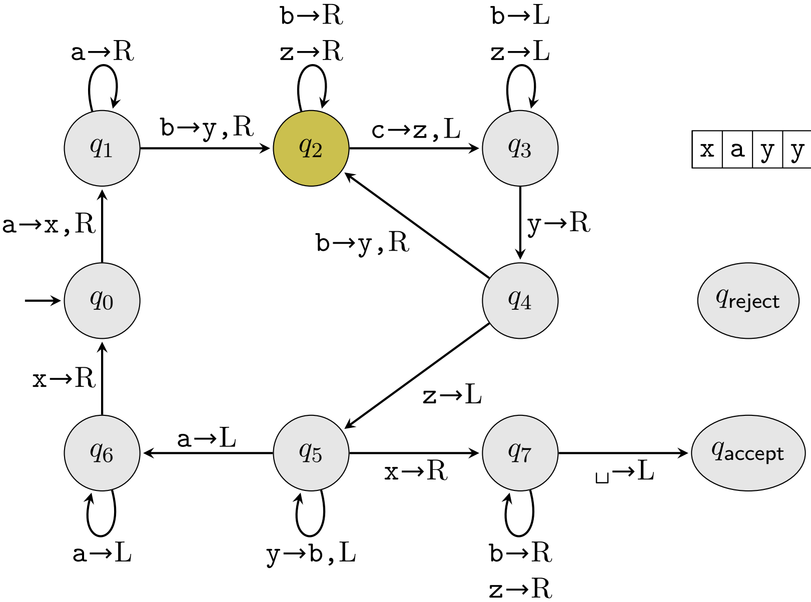
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ w = aabbbcccccc



x	a	y	y	b	z	c	c	c	c	c	
					↑						

Multiplication TM

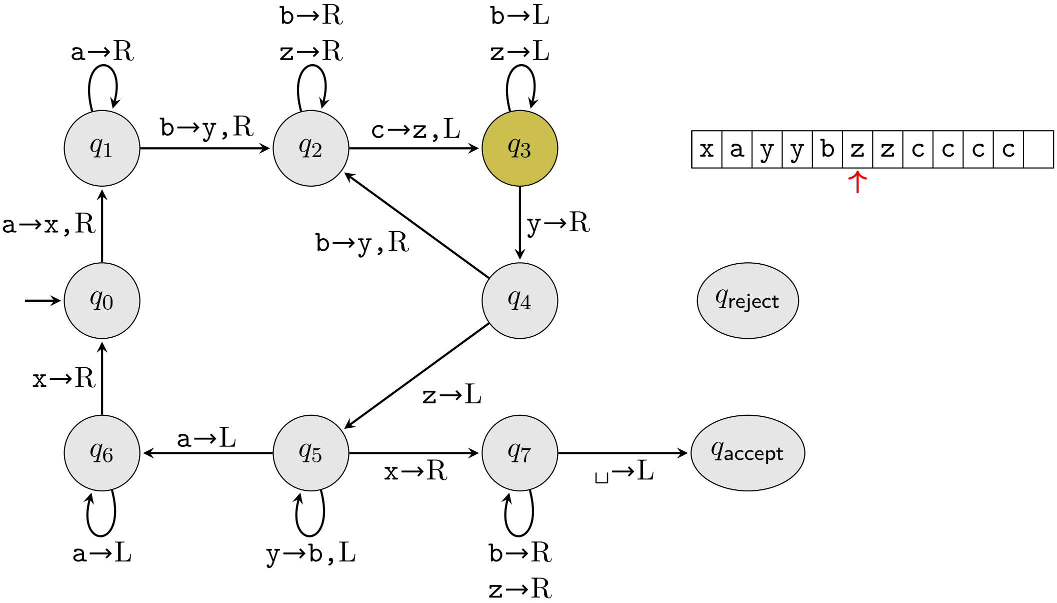
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	y	y	b	z	c	c	c	c	c	
						↑					

Multiplication TM

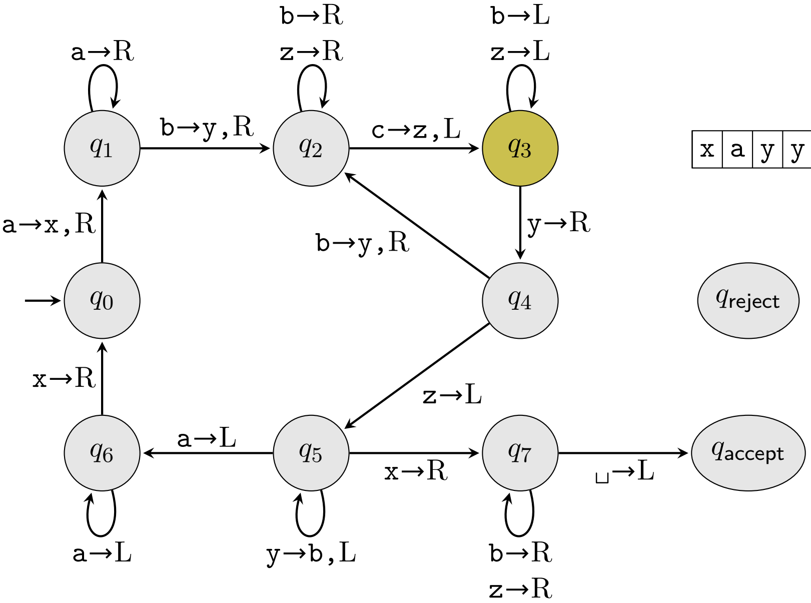
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	y	y	b	z	z	c	c	c	c	

Multiplication TM

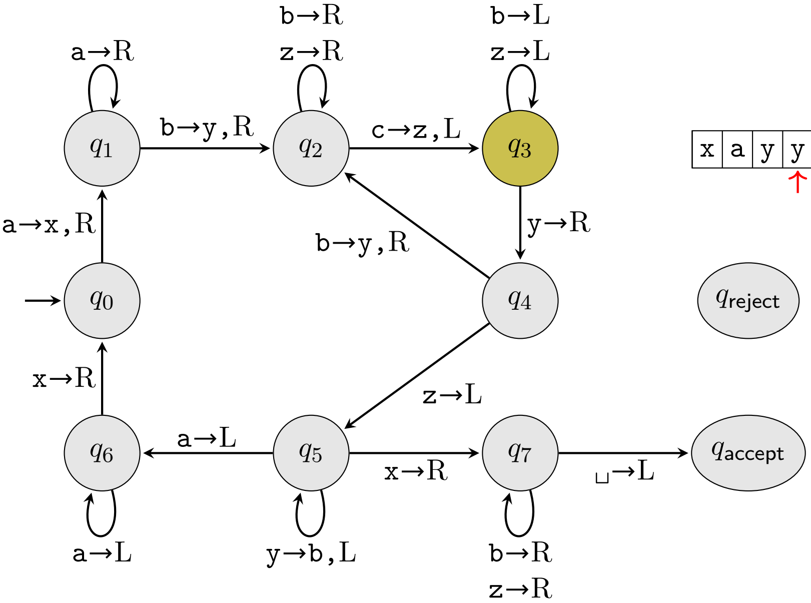
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	y	y	b	z	z	c	c	c	c	

Multiplication TM

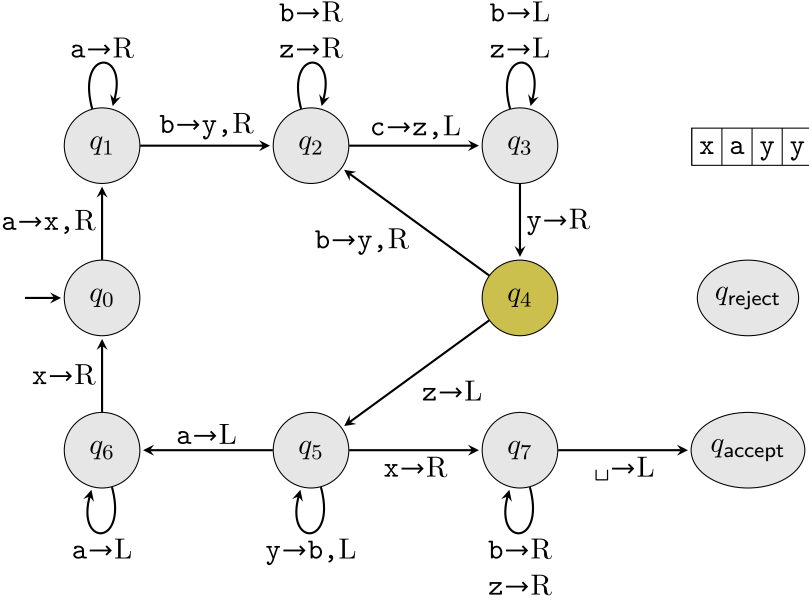
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i,j,k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	y	y	b	z	z	c	c	c	c	
			↑								

Multiplication TM

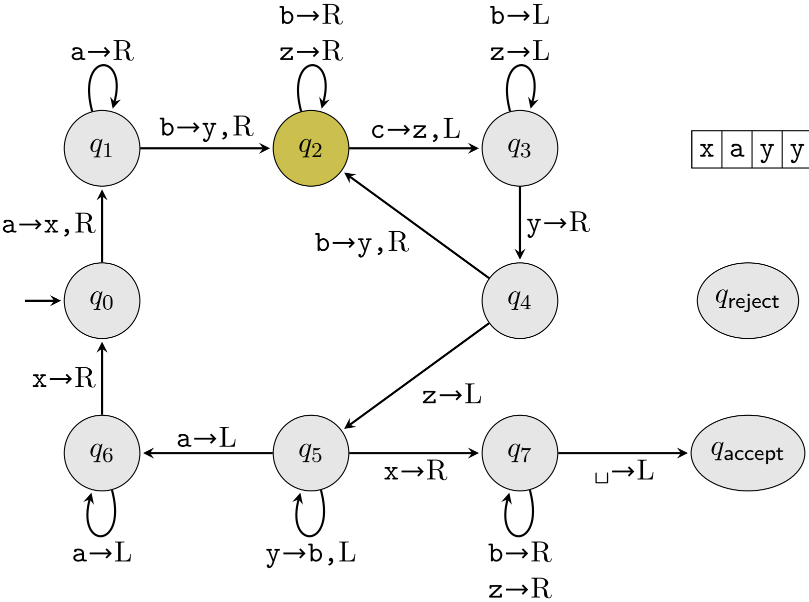
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	y	y	b	z	z	c	c	c	c	
				↑							

Multiplication TM

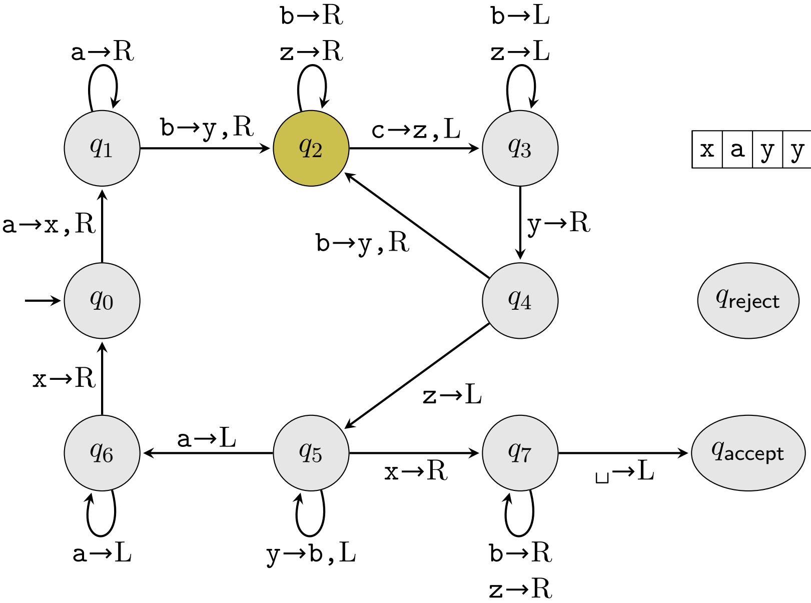
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	y	y	y	z	z	c	c	c	c	

Multiplication TM

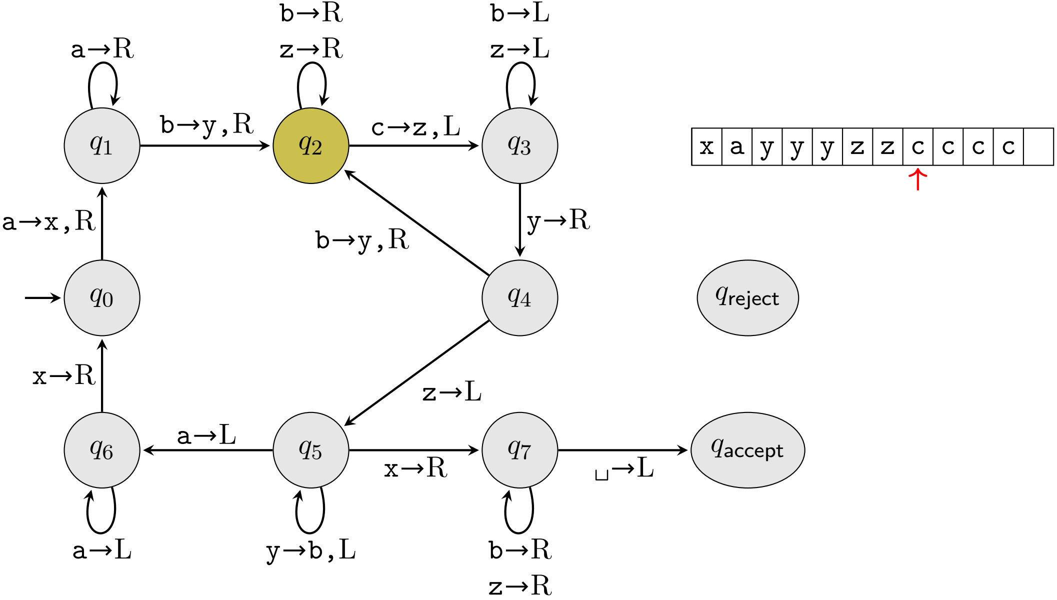
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \texttt{aabbbcccccc}$



x	a	y	y	y	z	z	c	c	c	c	
						↑					

Multiplication TM

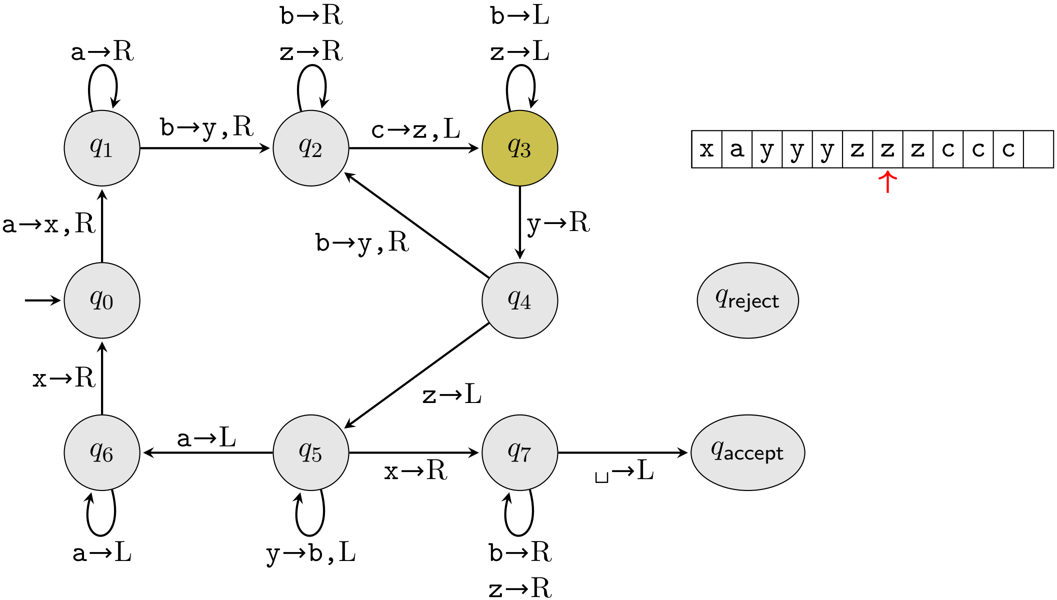
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	y	y	y	z	z	c	c	c	c	
							↑				

Multiplication TM

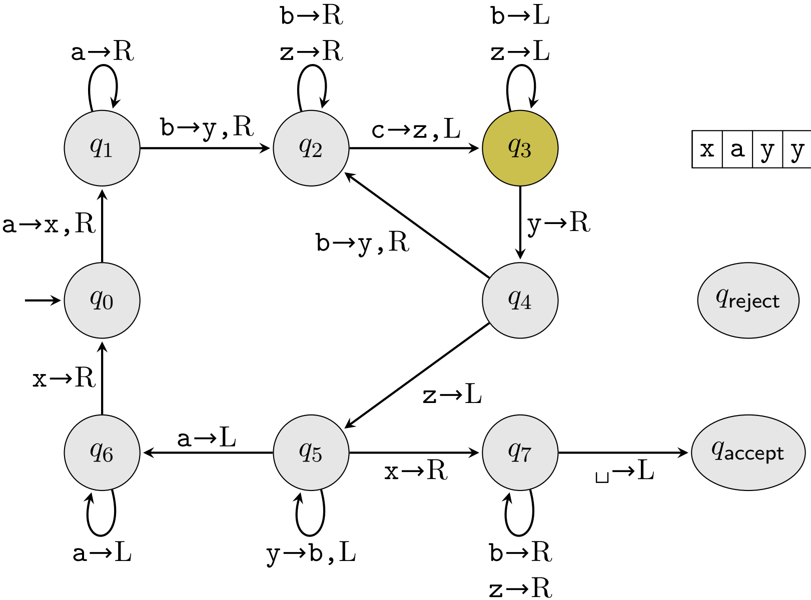
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	y	y	y	z	z	z	c	c	c	
						↑					

Multiplication TM

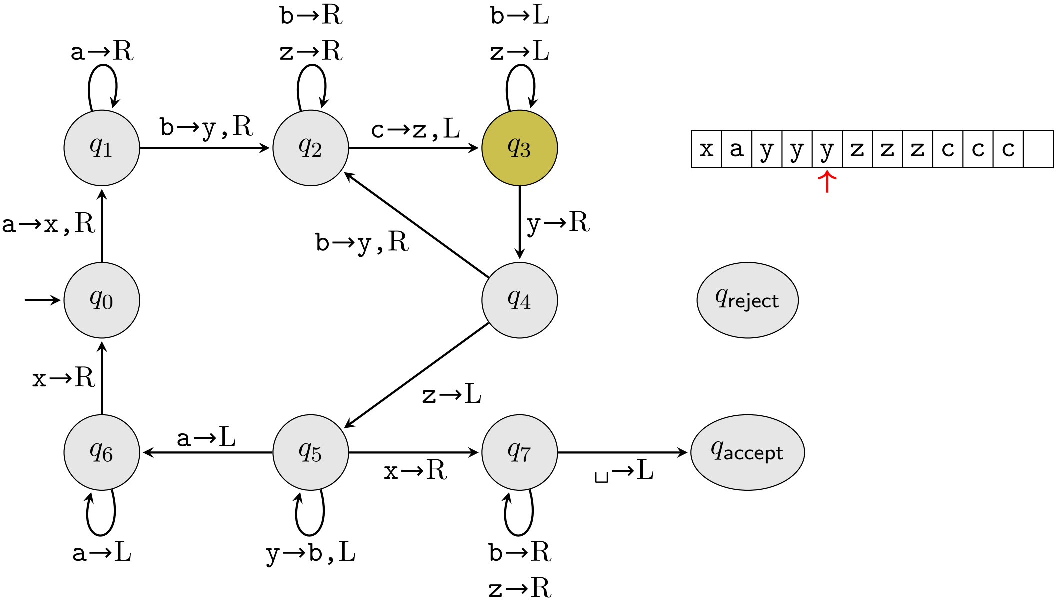
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \texttt{aabbbcccccc}$



x	a	y	y	y	z	z	z	c	c	c	
					↑						

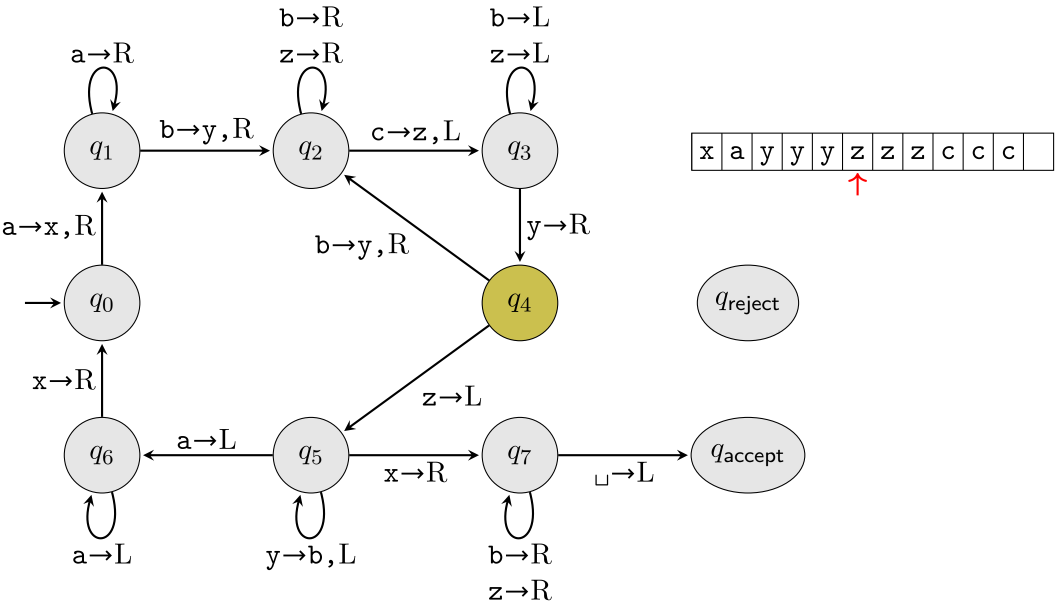
Multiplication TM

$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \mathtt{aabbbcccccc}$



Multiplication TM

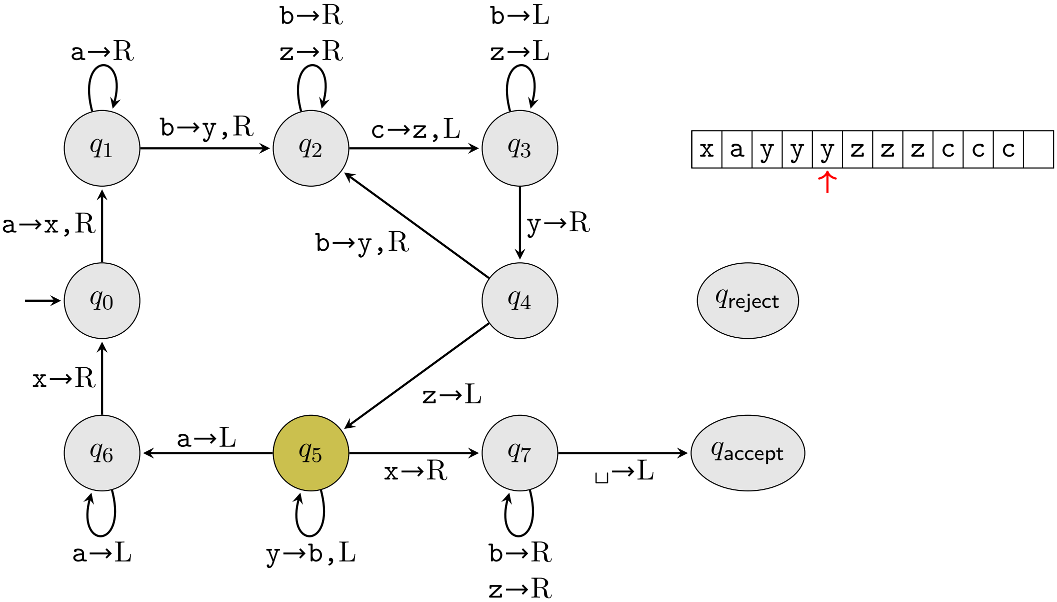
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	y	y	y	z	z	z	c	c	c	

Multiplication TM

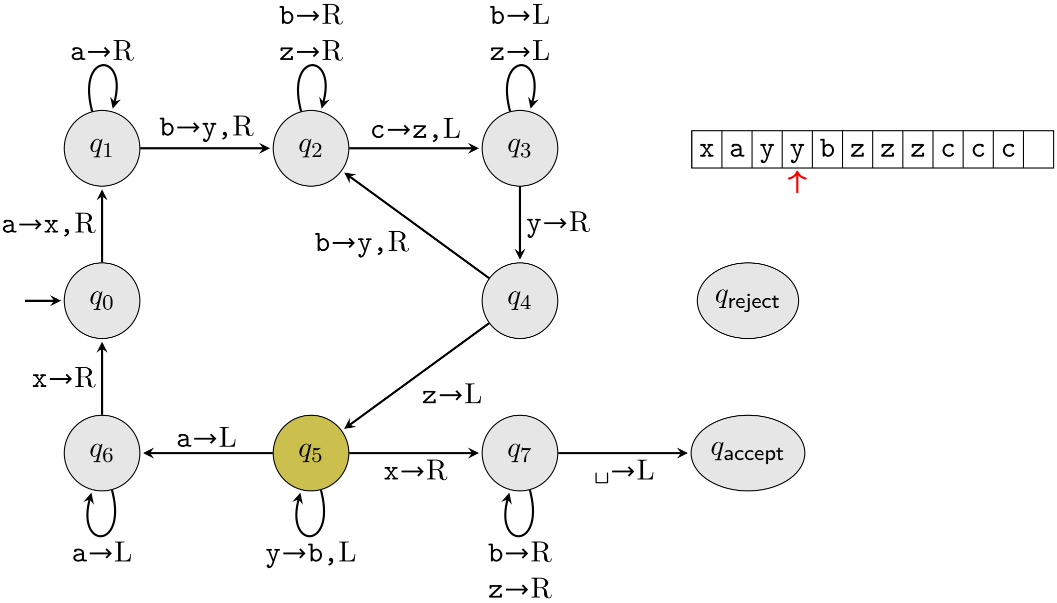
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	y	y	y	z	z	z	c	c	c	

Multiplication TM

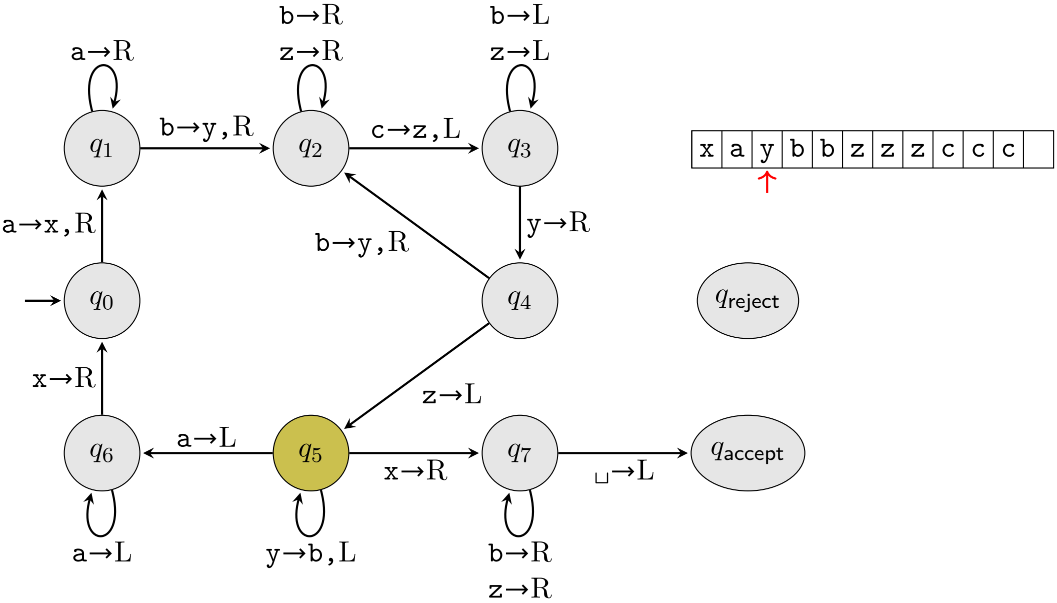
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \texttt{aabbbcccccc}$



x	a	y	y	b	z	z	z	c	c	c	

Multiplication TM

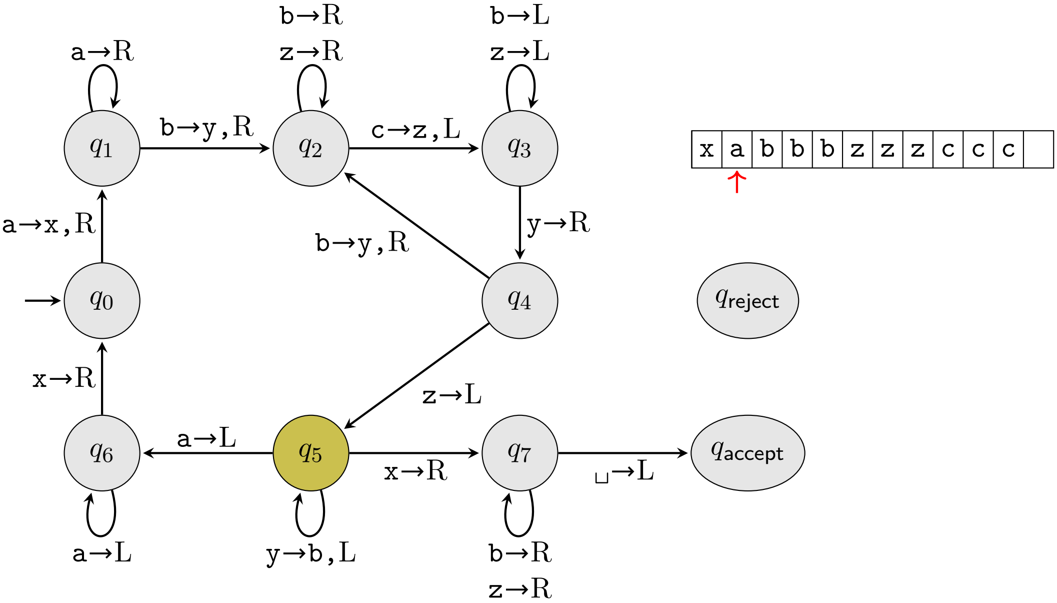
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	y	b	b	z	z	z	c	c	c	

Multiplication TM

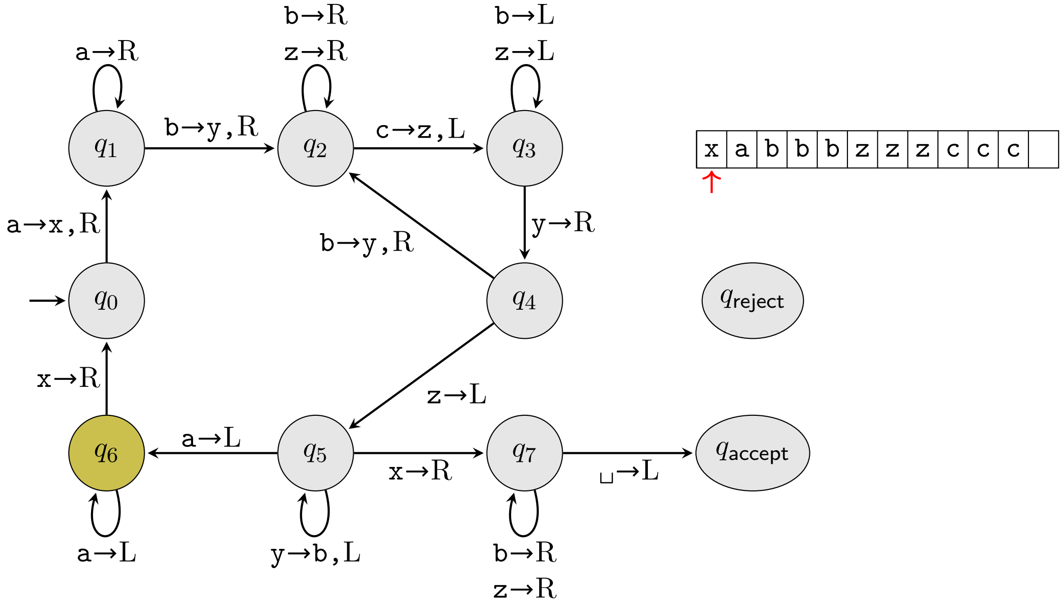
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	a	b	b	b	z	z	z	c	c	c	
	↑										

Multiplication TM

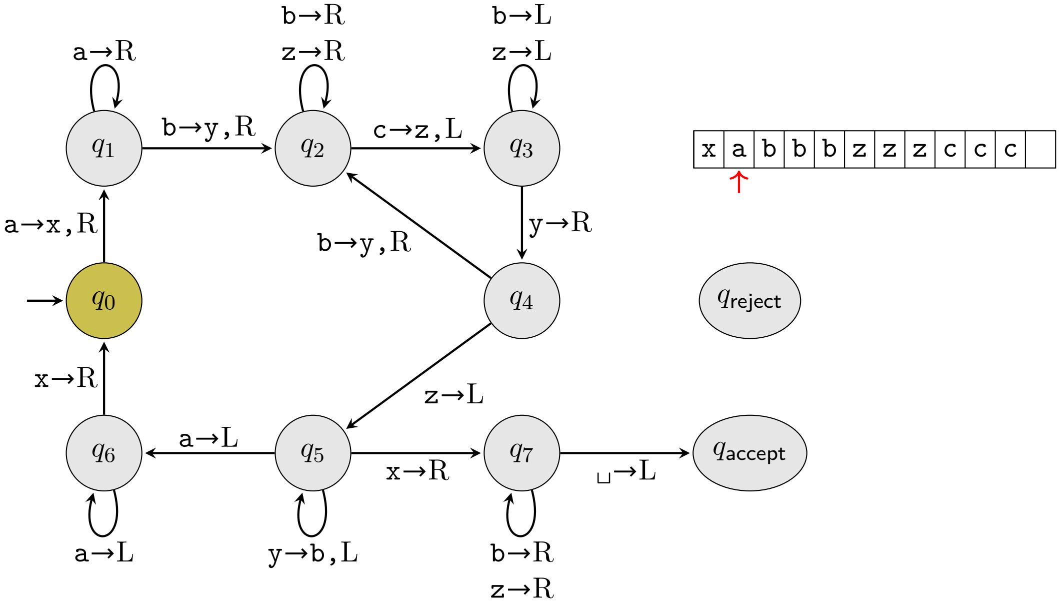
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \texttt{aabbbcccccc}$



x	a	b	b	b	z	z	z	c	c	c	

Multiplication TM

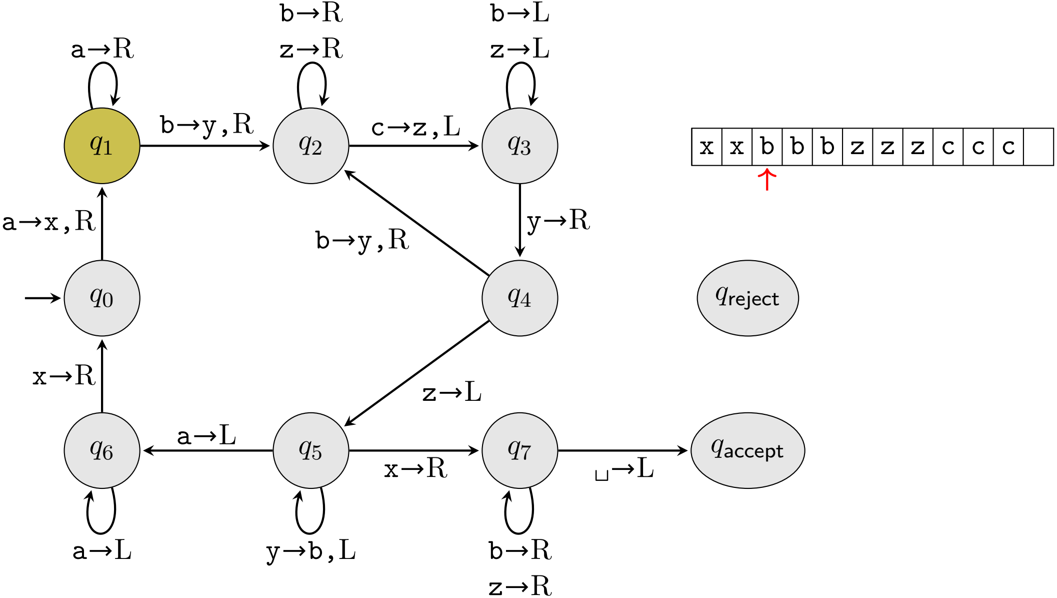
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \texttt{aabbbcccccc}$



x	a	b	b	b	z	z	z	c	c	c	

Multiplication TM

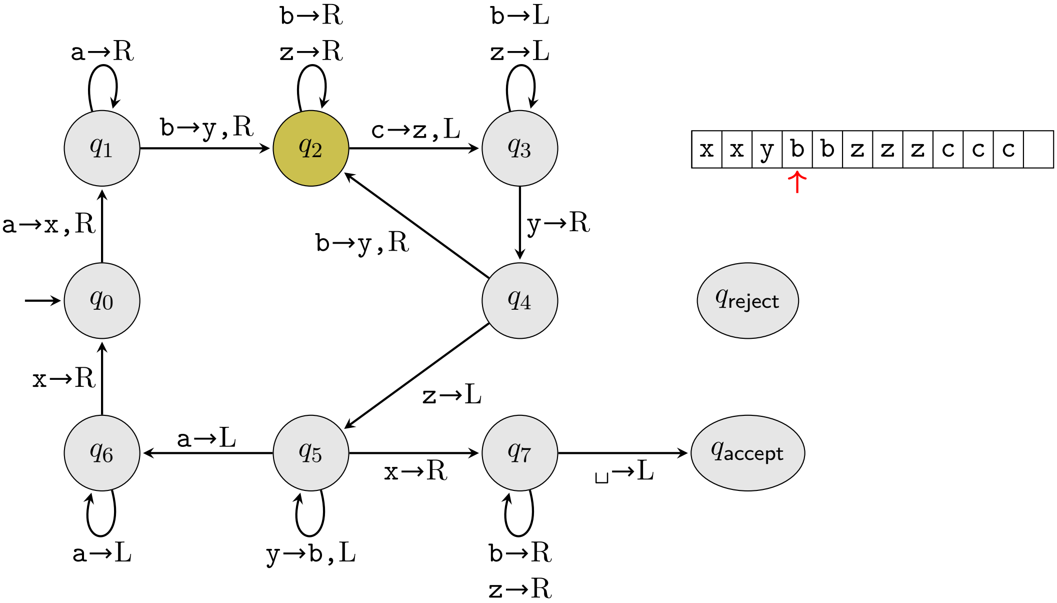
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \texttt{aabbbcccccc}$



x	x	b	b	b	z	z	z	c	c	c	
		↑									

Multiplication TM

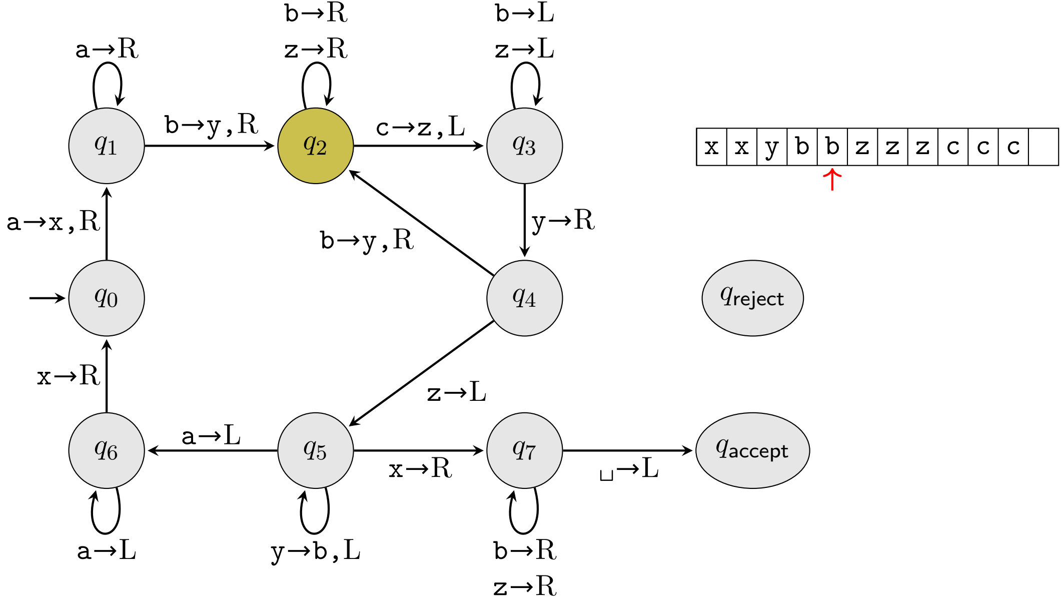
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \texttt{aabbbcccccc}$



x	x	y	b	b	z	z	z	c	c	c	
			↑								

Multiplication TM

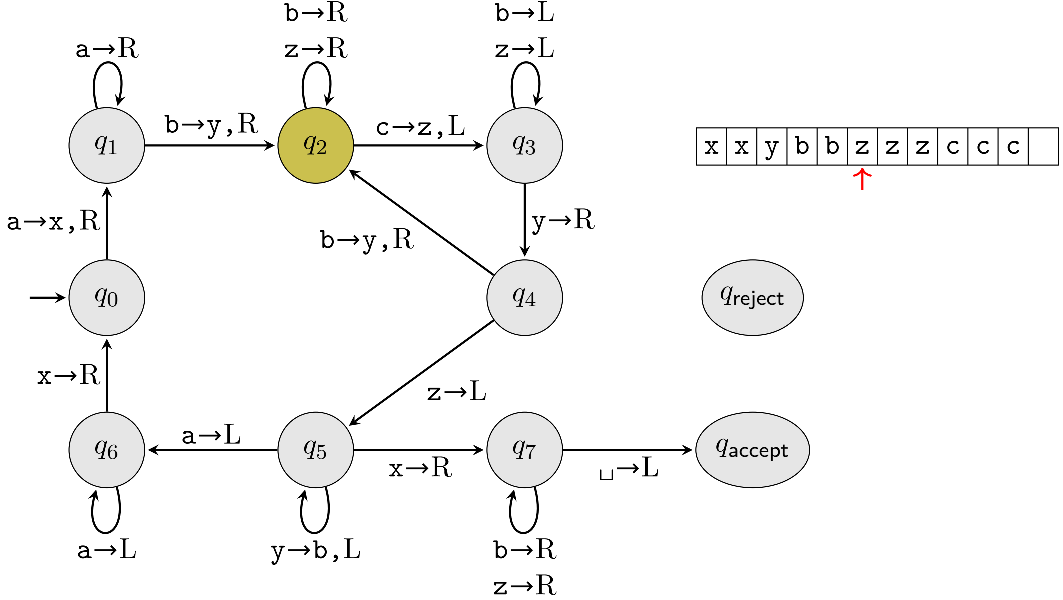
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \mathtt{aabbbcccccc}$



x	x	y	b	b	z	z	z	c	c	c	

Multiplication TM

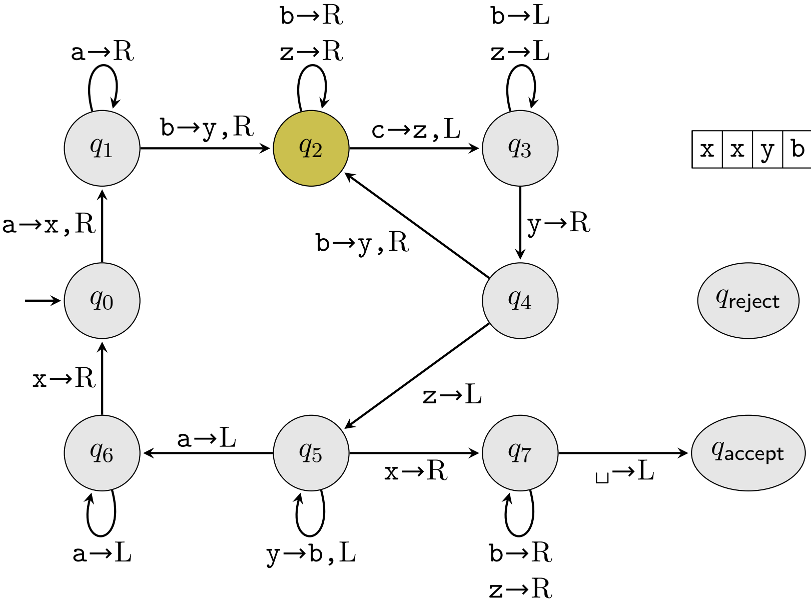
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \texttt{aabbbcccccc}$



x	x	y	b	b	z	z	z	c	c	c	
					↑						

Multiplication TM

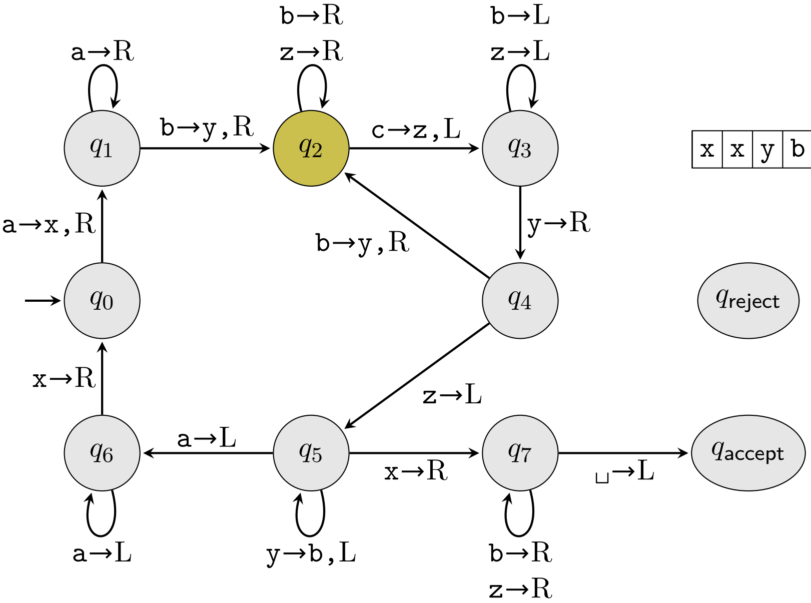
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	b	b	z	z	z	c	c	c	

Multiplication TM

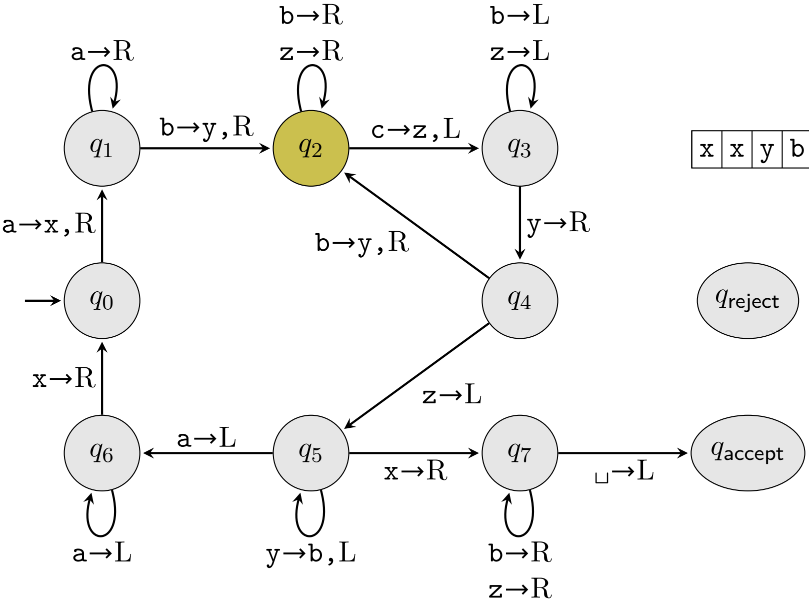
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	b	b	z	z	z	c	c	c	

Multiplication TM

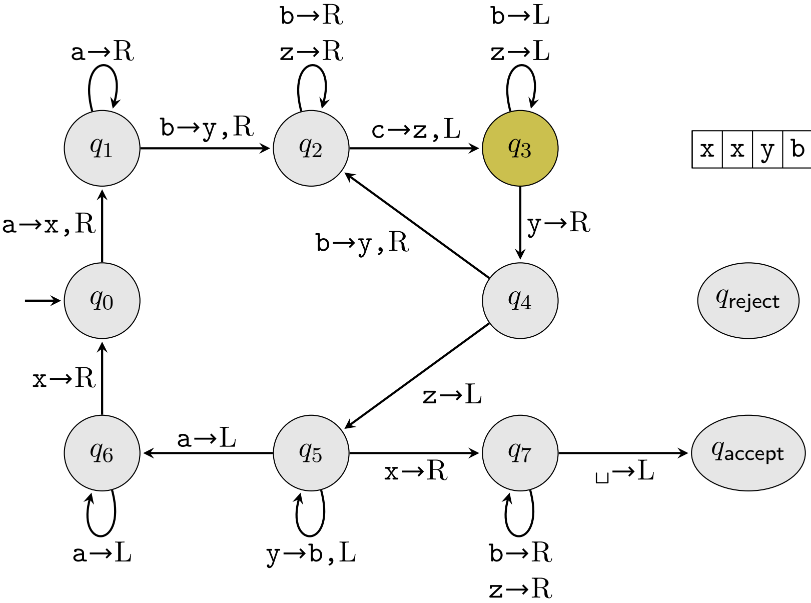
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	b	b	z	z	z	c	c	c	
								↑			

Multiplication TM

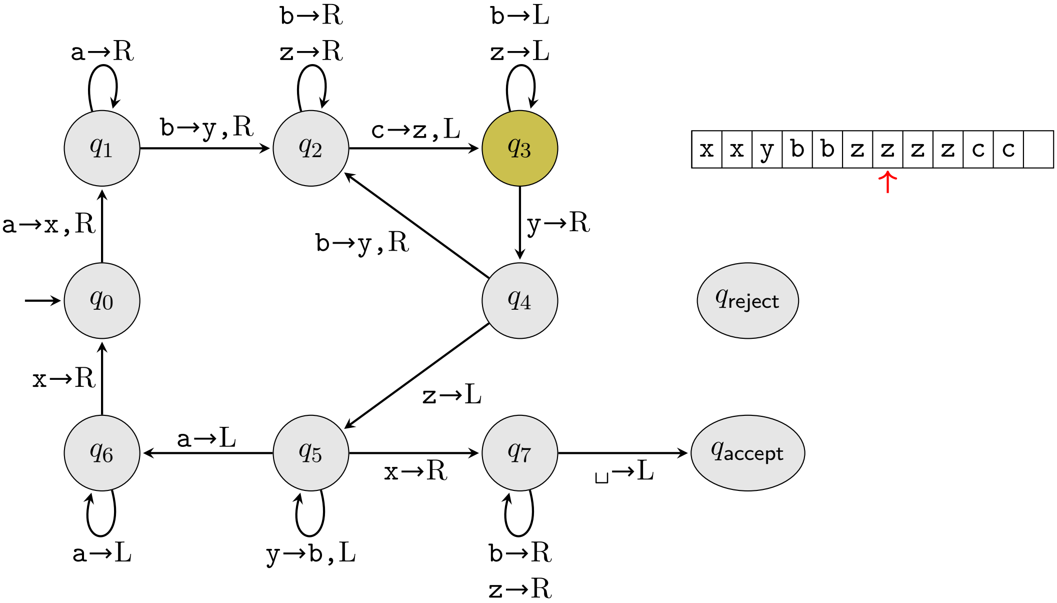
$B = \{\mathtt{a}^i \mathtt{b}^j \mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	b	b	z	z	z	z	c	c	
							↑				

Multiplication TM

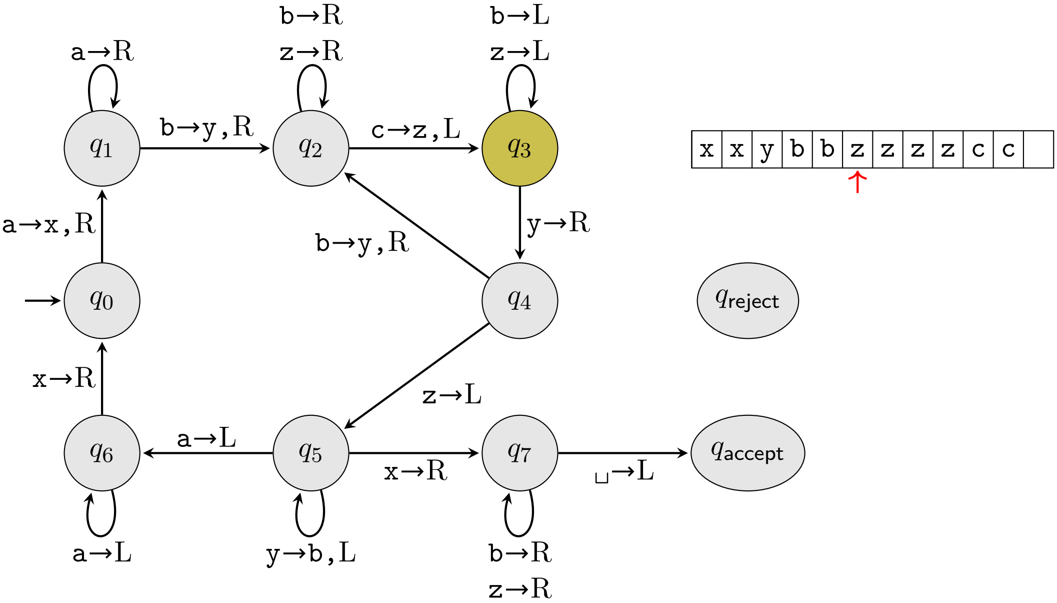
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	b	b	z	z	z	z	c	c	

Multiplication TM

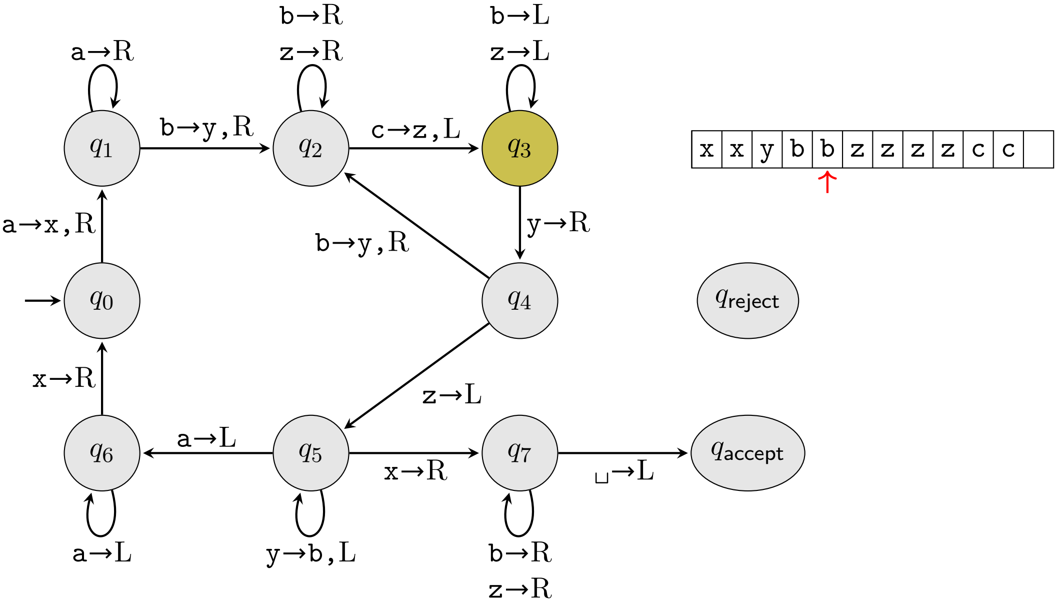
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	b	b	z	z	z	z	c	c	
					↑						

Multiplication TM

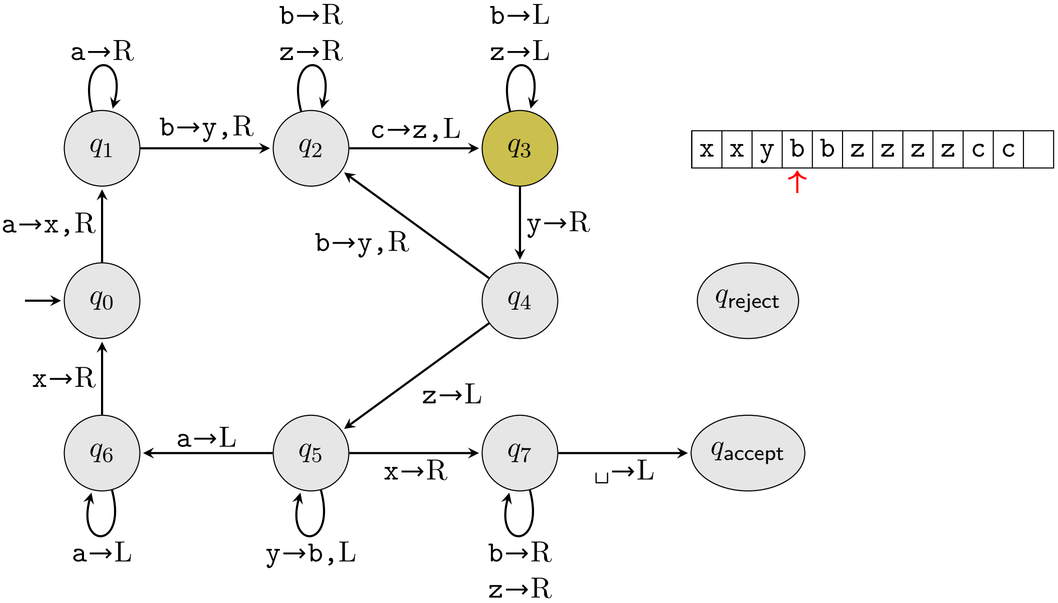
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	b	b	z	z	z	z	c	c	

Multiplication TM

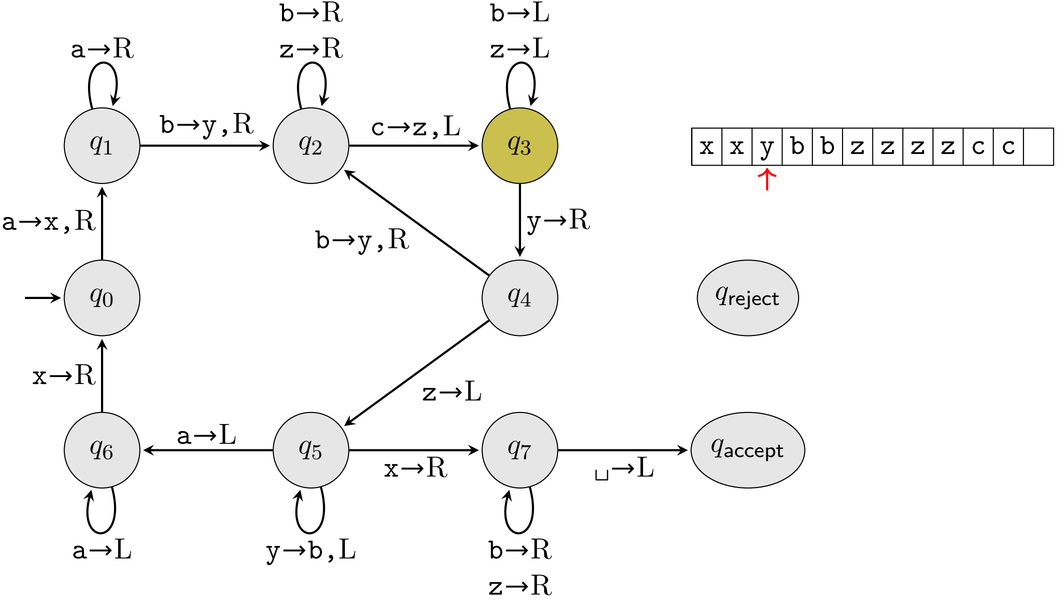
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	b	b	z	z	z	z	c	c	

Multiplication TM

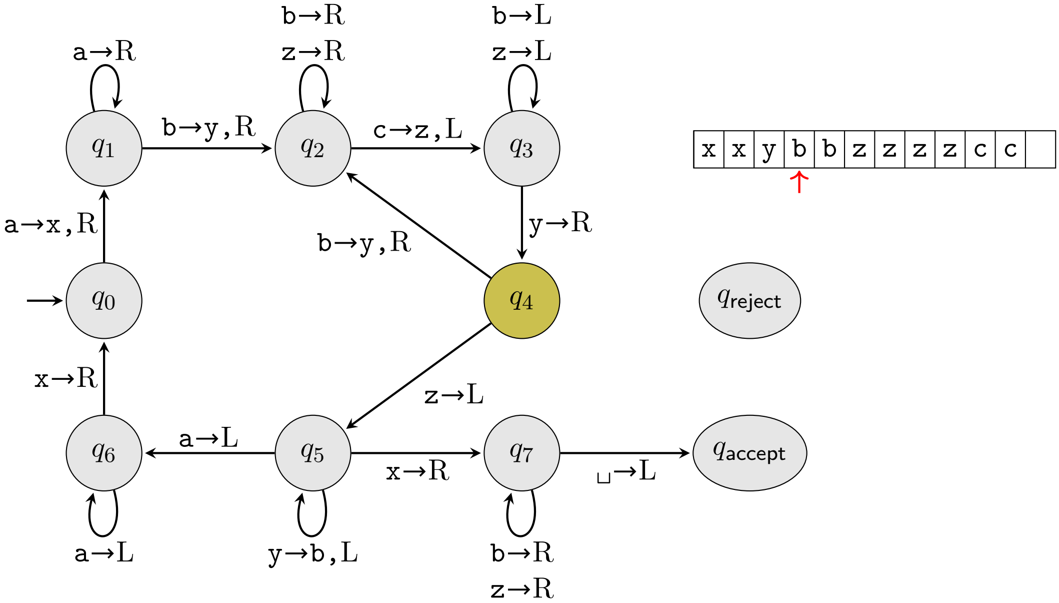
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	b	b	z	z	z	z	c	c	

Multiplication TM

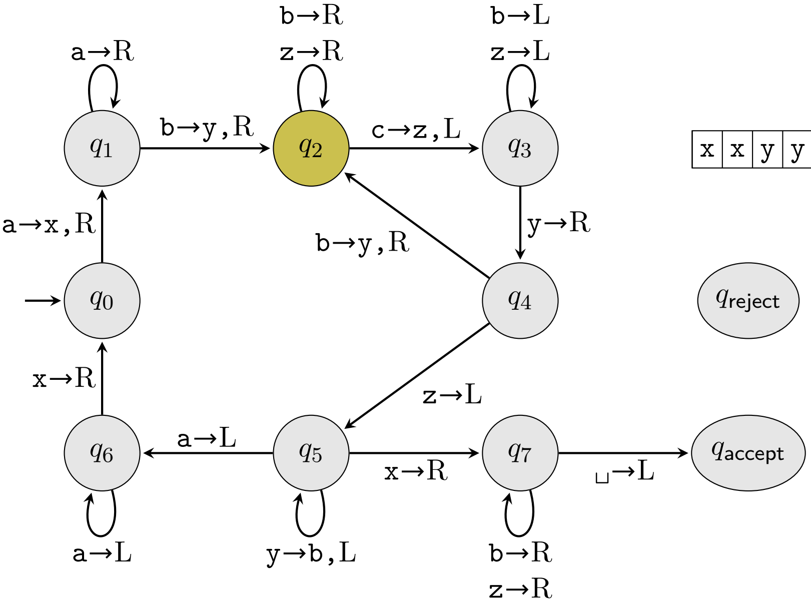
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \mathtt{aabbbcccccc}$



x	x	y	b	b	z	z	z	z	c	c	

Multiplication TM

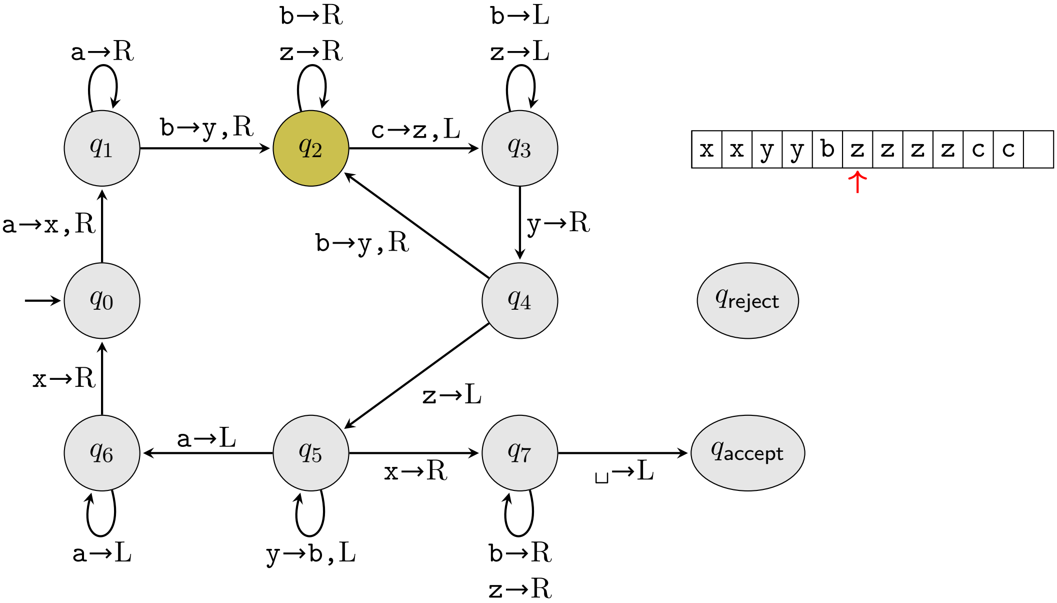
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	b	z	z	z	z	c	c	

Multiplication TM

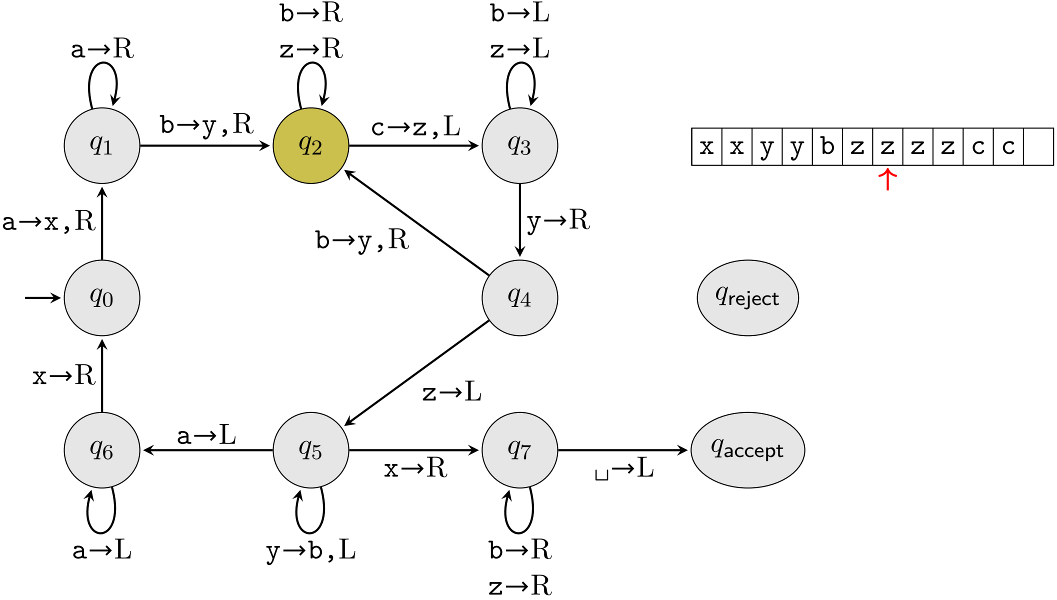
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \mathtt{aabbbcccccc}$



x	x	y	y	b	z	z	z	z	c	c	

Multiplication TM

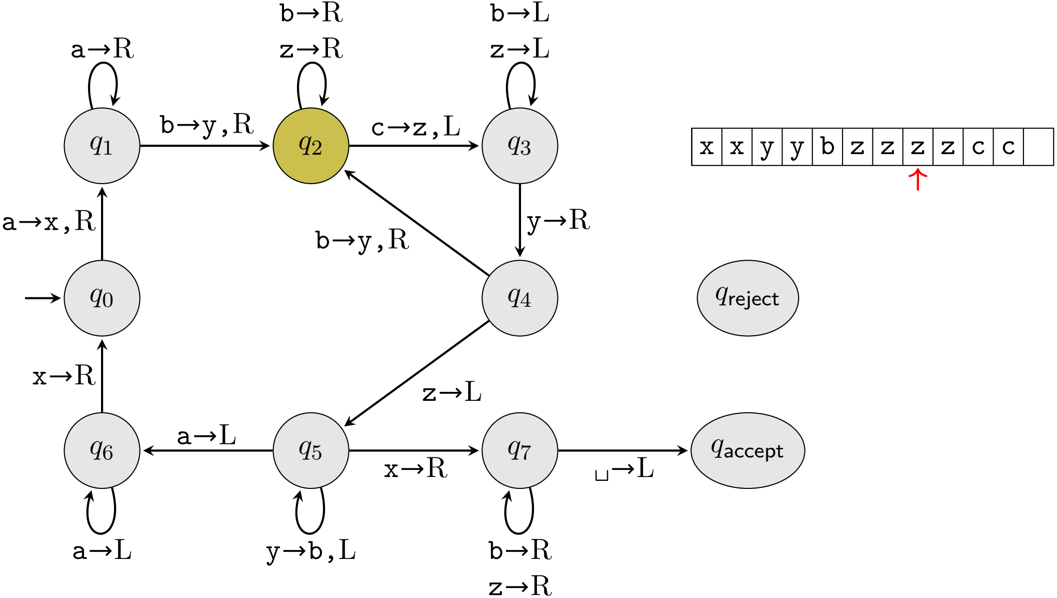
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	b	z	z	z	z	c	c	

Multiplication TM

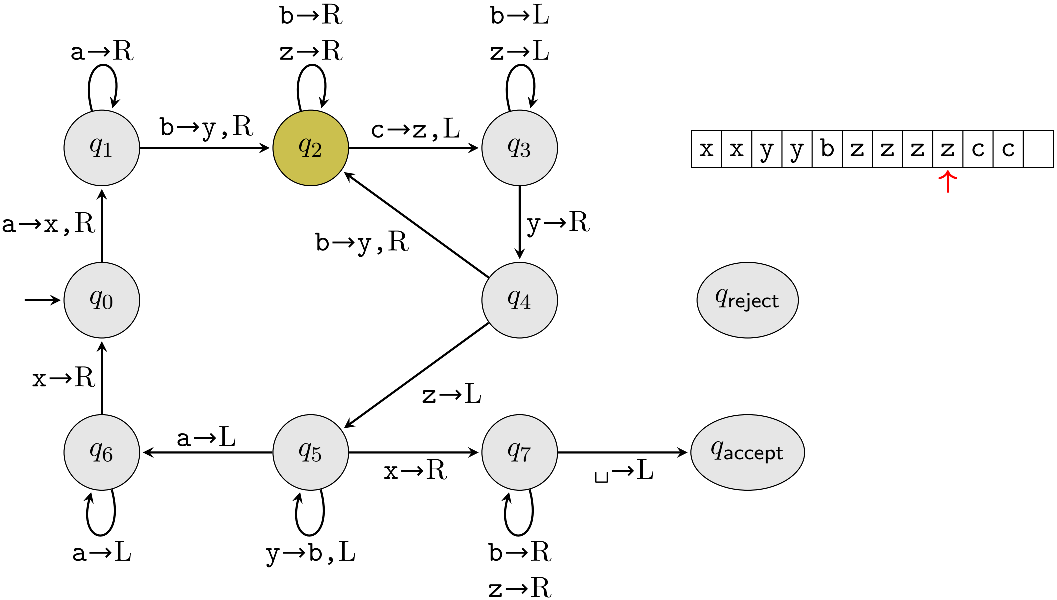
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \texttt{aabbbcccccc}$



x	x	y	y	b	z	z	z	z	c	c	

Multiplication TM

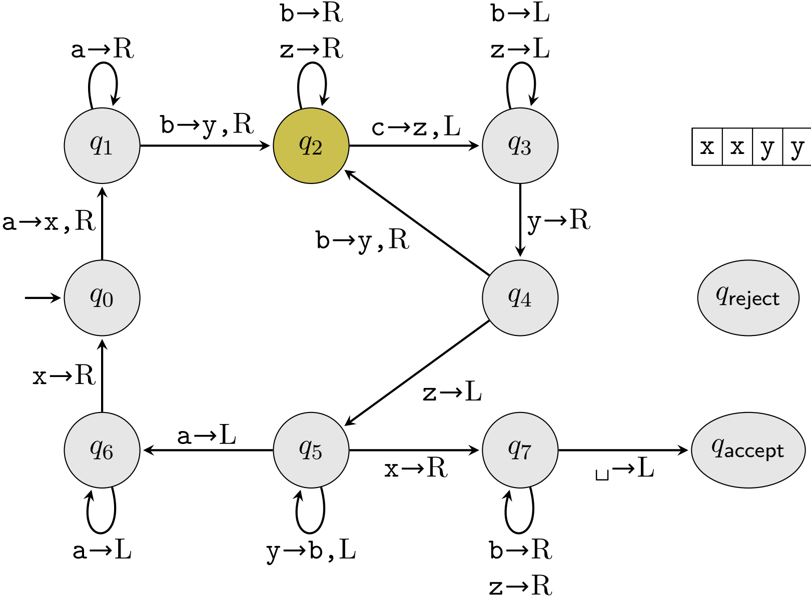
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	b	z	z	z	z	c	c	

Multiplication TM

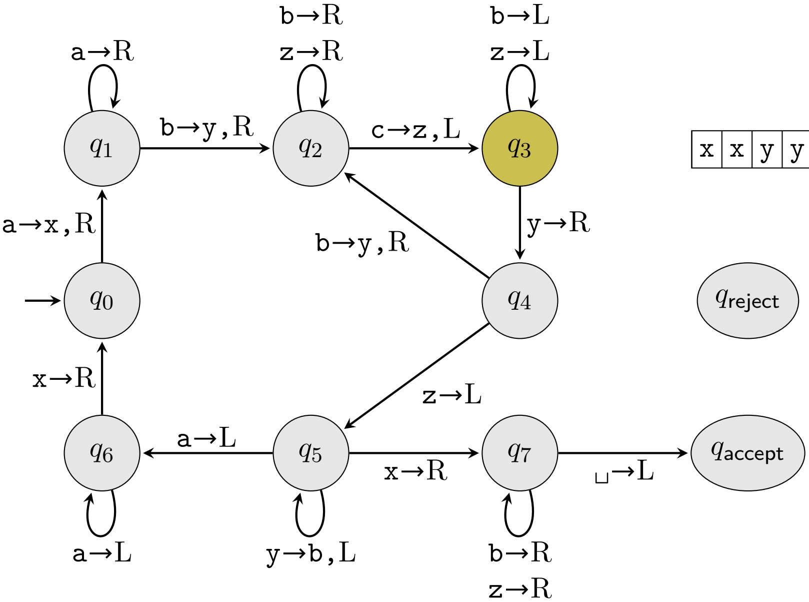
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \texttt{aabbbcccccc}$



x	x	y	y	b	z	z	z	z	c	c	

Multiplication TM

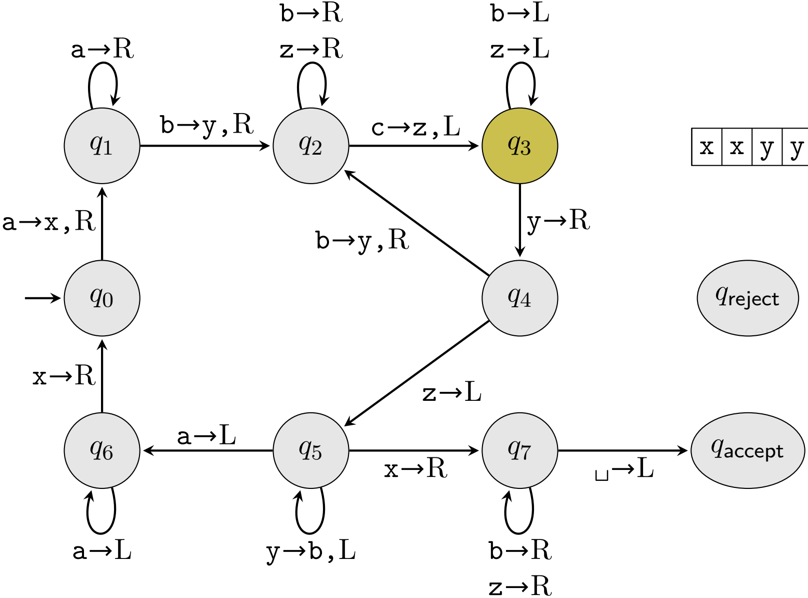
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \texttt{aabbbcccccc}$



x	x	y	y	b	z	z	z	z	z	c	
								↑			

Multiplication TM

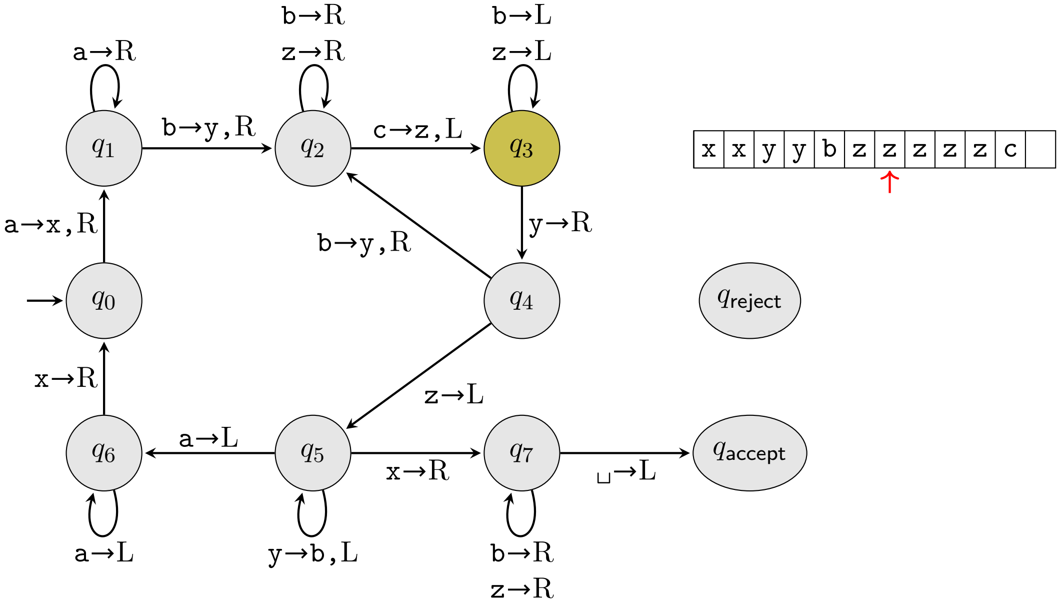
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	b	z	z	z	z	z	c	

Multiplication TM

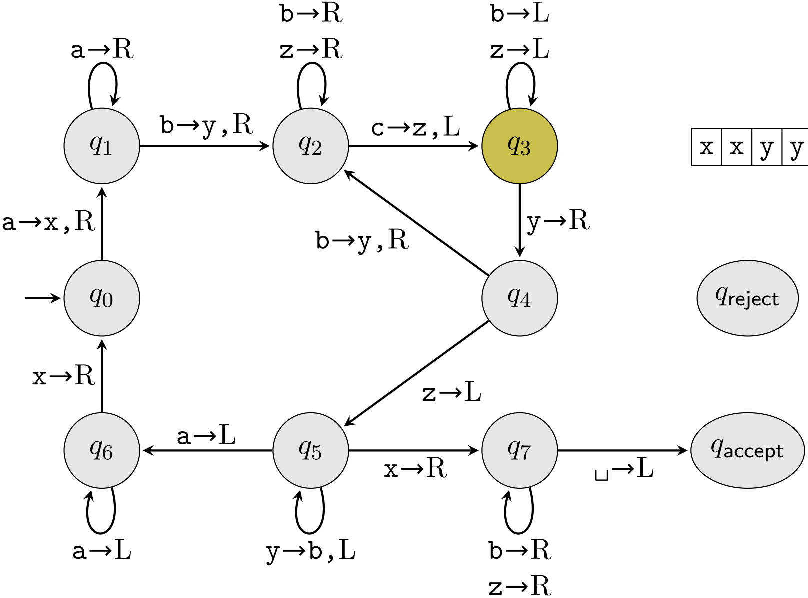
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	b	z	z	z	z	z	c	

Multiplication TM

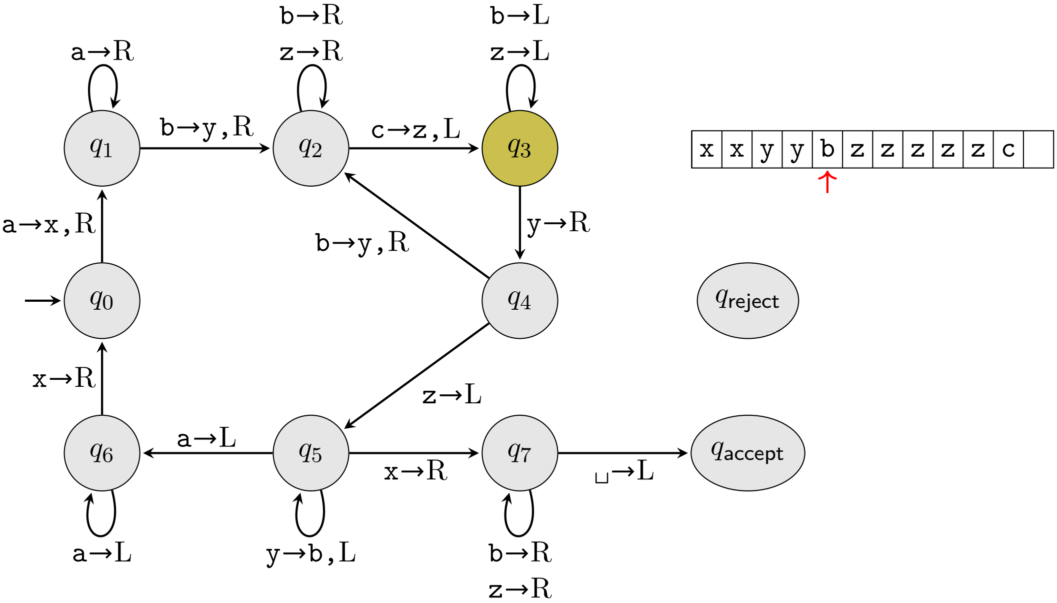
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	b	z	z	z	z	z	c	
					↑						

Multiplication TM

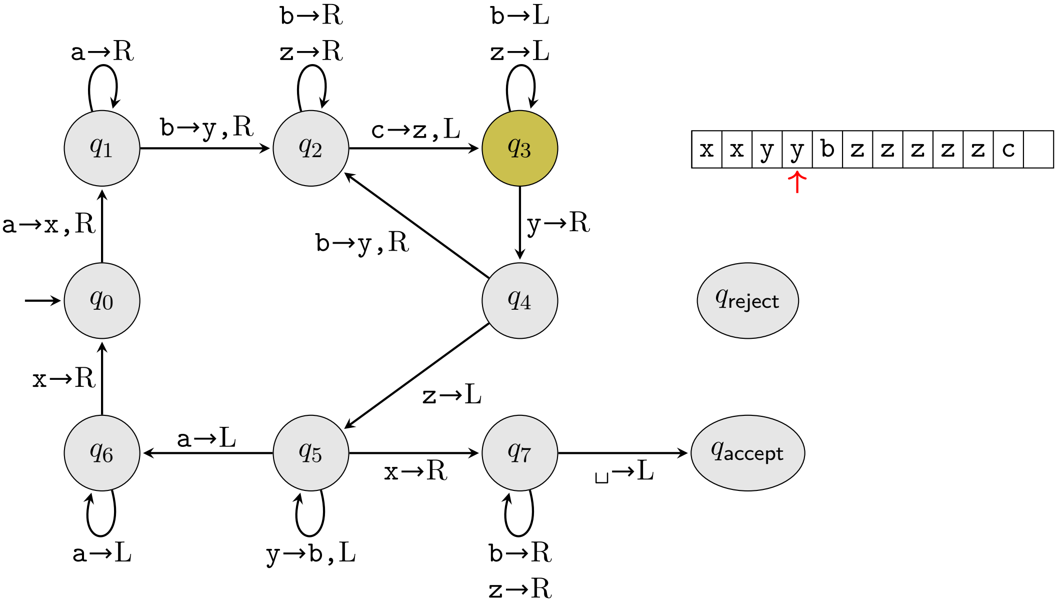
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	b	z	z	z	z	z	c	
				↑							

Multiplication TM

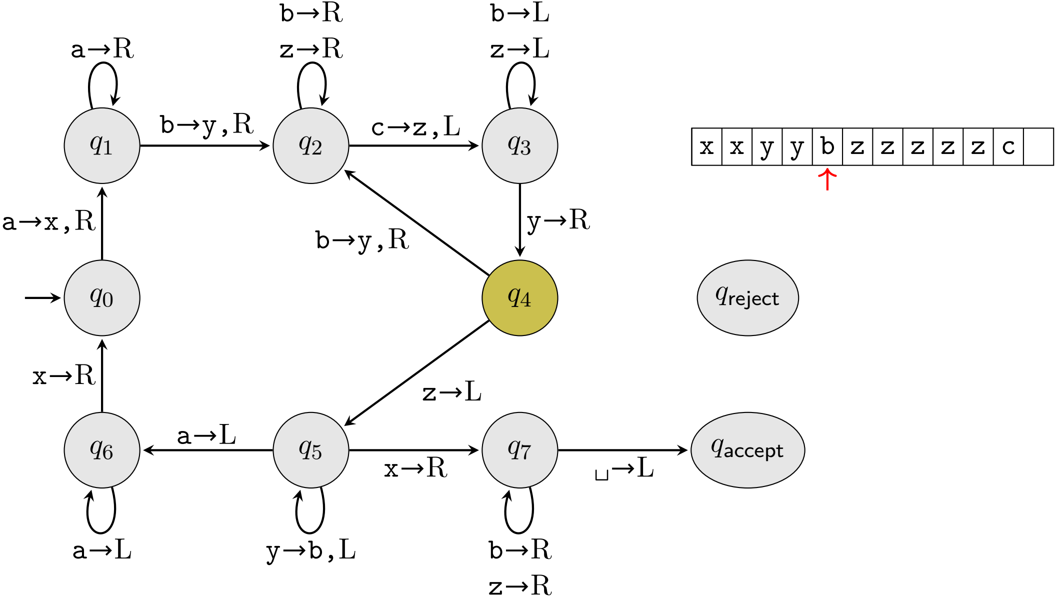
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \texttt{aabbbcccccc}$



x	x	y	y	b	z	z	z	z	z	c	

Multiplication TM

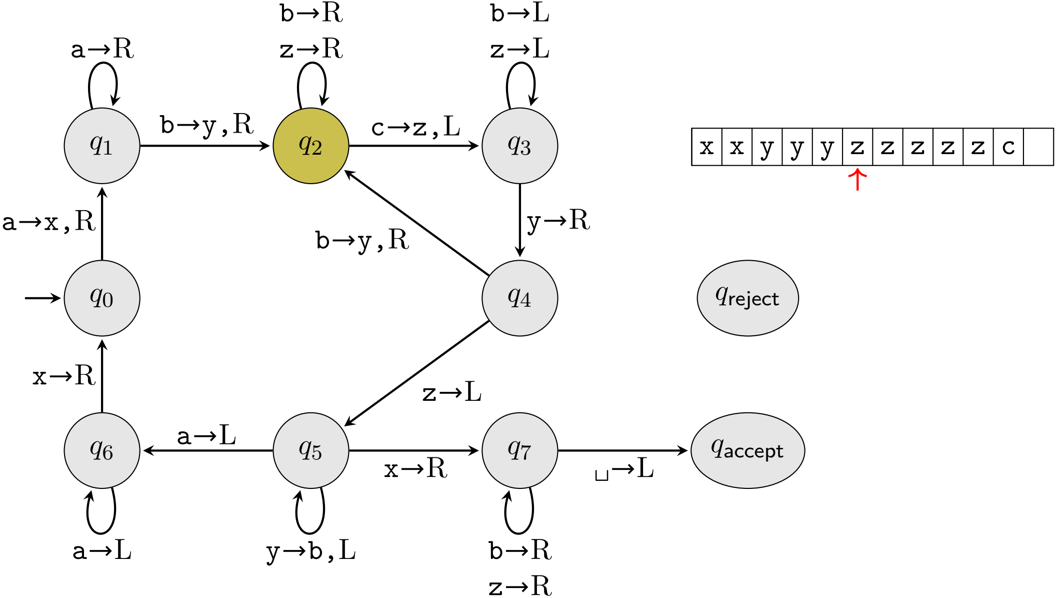
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	b	z	z	z	z	z	c	

Multiplication TM

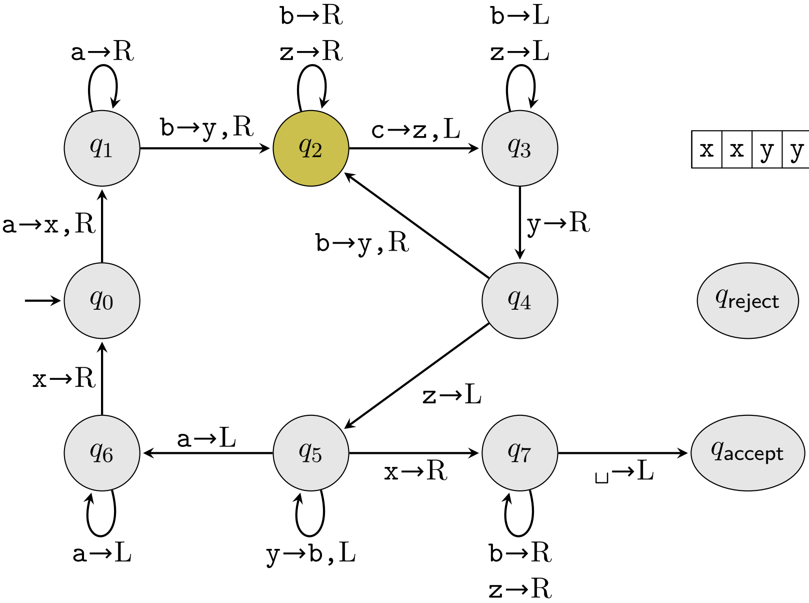
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \mathtt{aabbbcccccc}$



x	x	y	y	y	z	z	z	z	z	c	

Multiplication TM

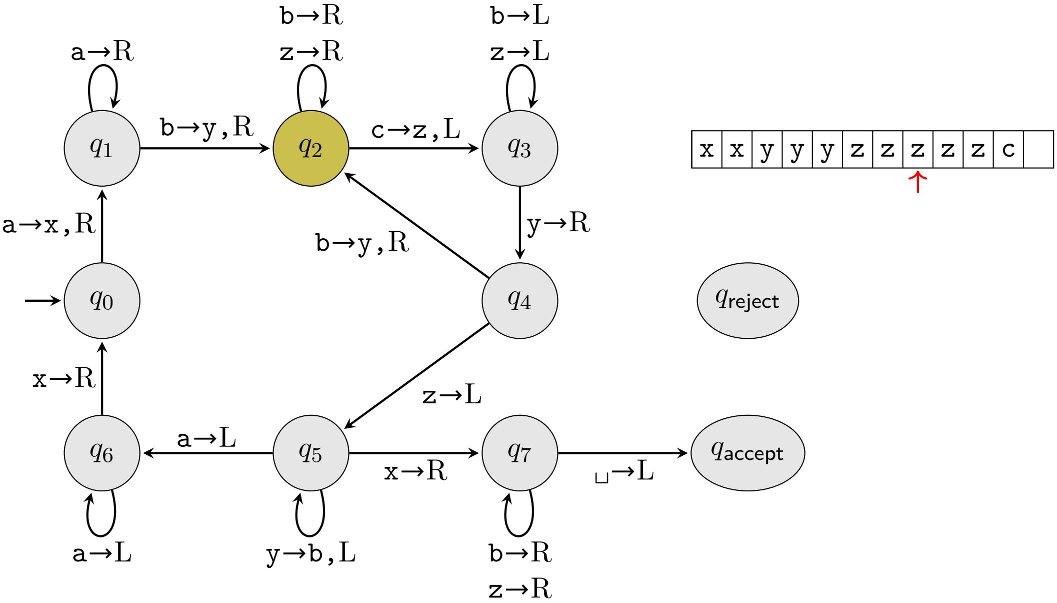
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \mathtt{aabbbcccccc}$



x	x	y	y	y	z	z	z	z	z	c	

Multiplication TM

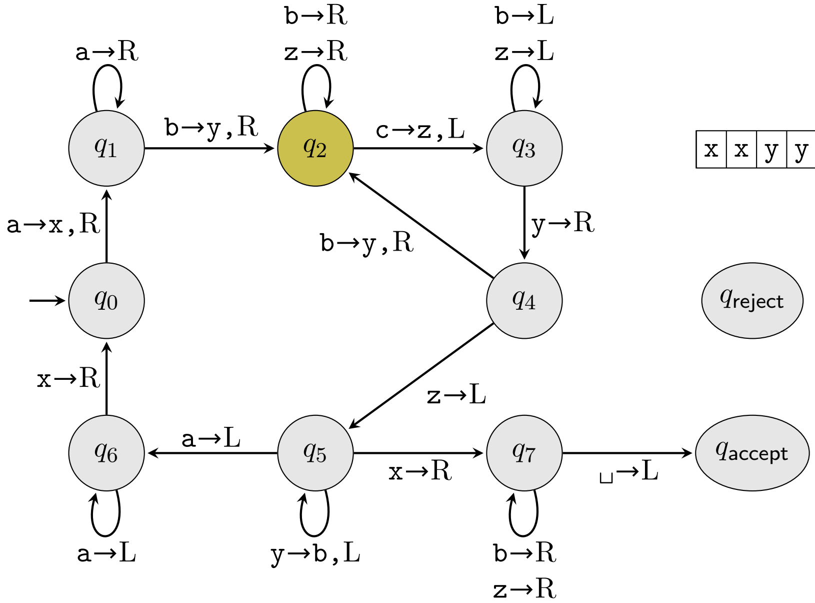
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	y	z	z	z	z	z	c	
							↑				

Multiplication TM

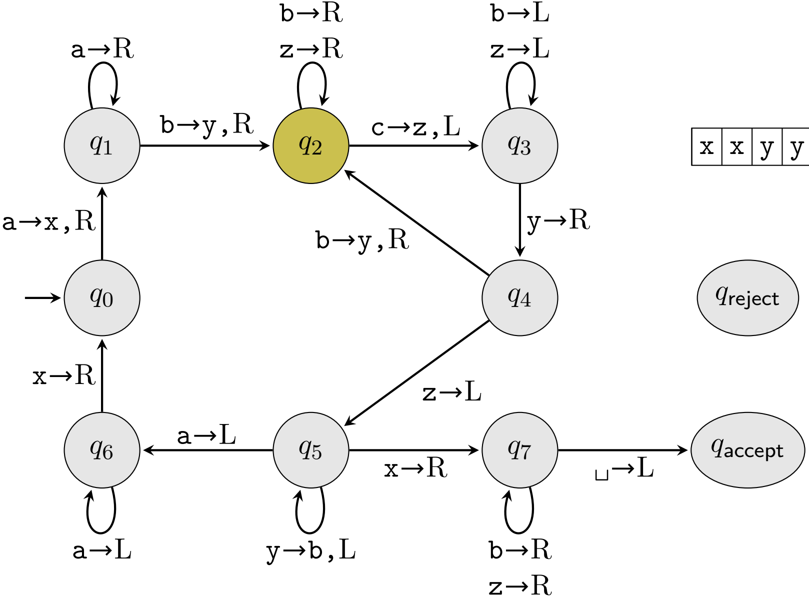
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	y	z	z	z	z	z	c	

Multiplication TM

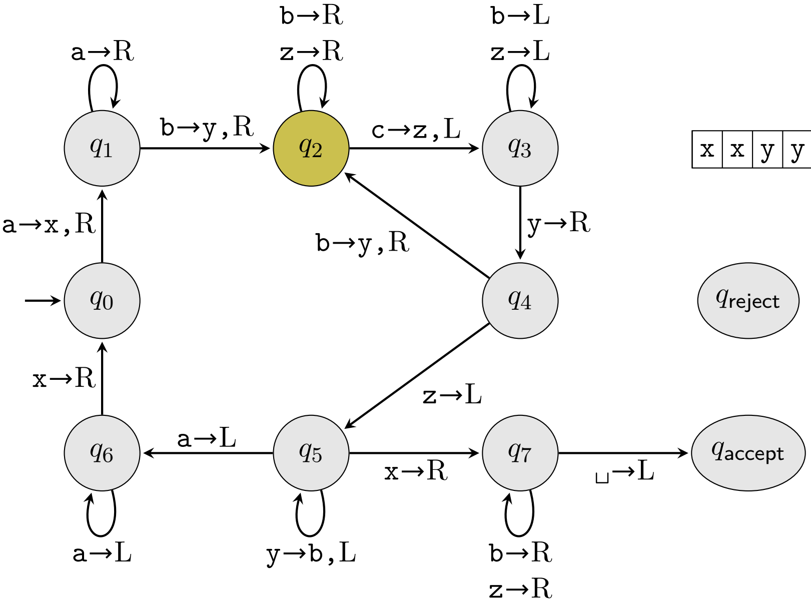
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \texttt{aabbbcccccc}$



x	x	y	y	y	z	z	z	z	z	c	
									↑		

Multiplication TM

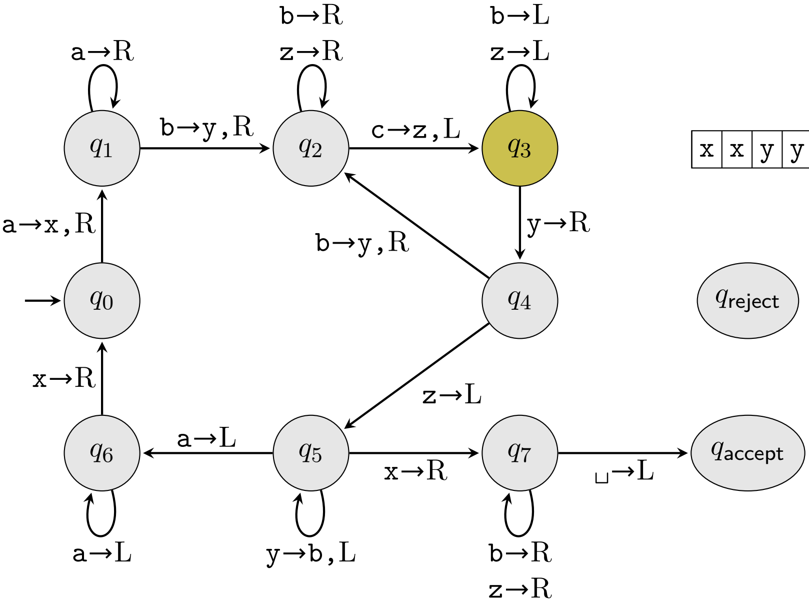
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	y	z	z	z	z	z	c	

Multiplication TM

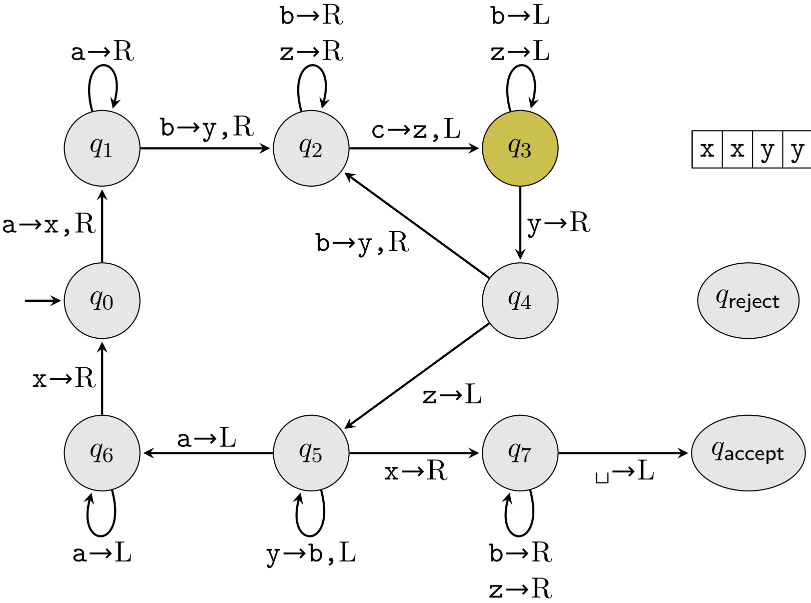
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \texttt{aabbbcccccc}$



x	x	y	y	y	z	z	z	z	z	z	
									↑		

Multiplication TM

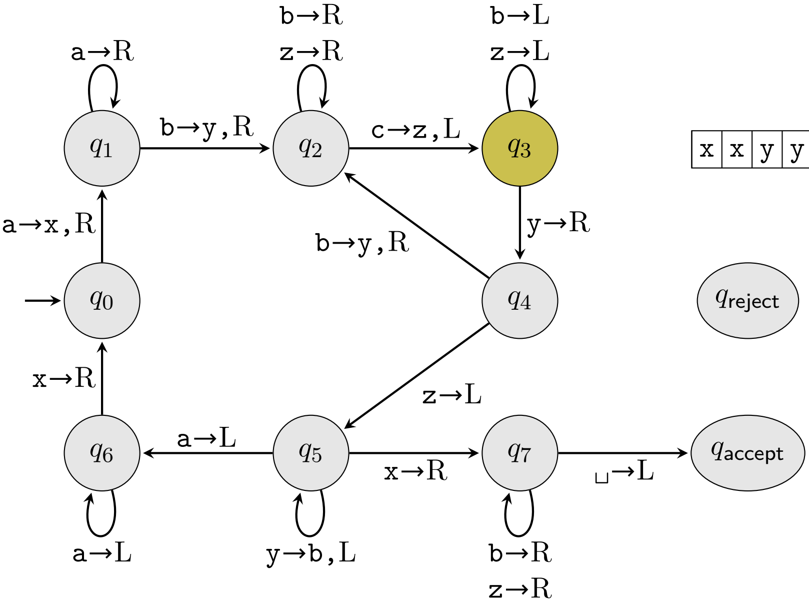
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	y	z	z	z	z	z	z	

Multiplication TM

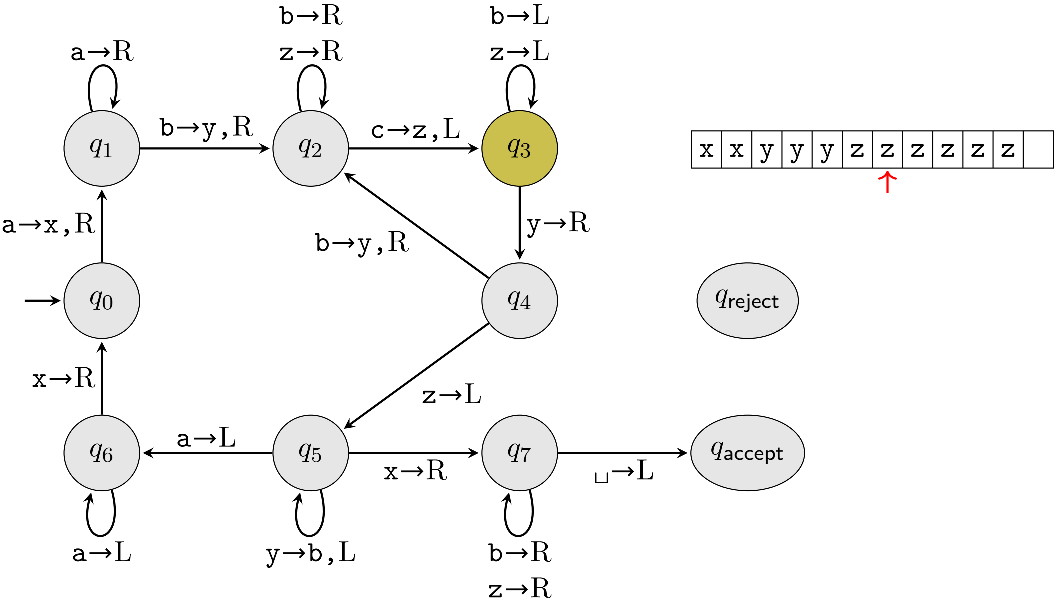
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \mathtt{aabbbcccccc}$



x	x	y	y	y	z	z	z	z	z	z	
							↑				

Multiplication TM

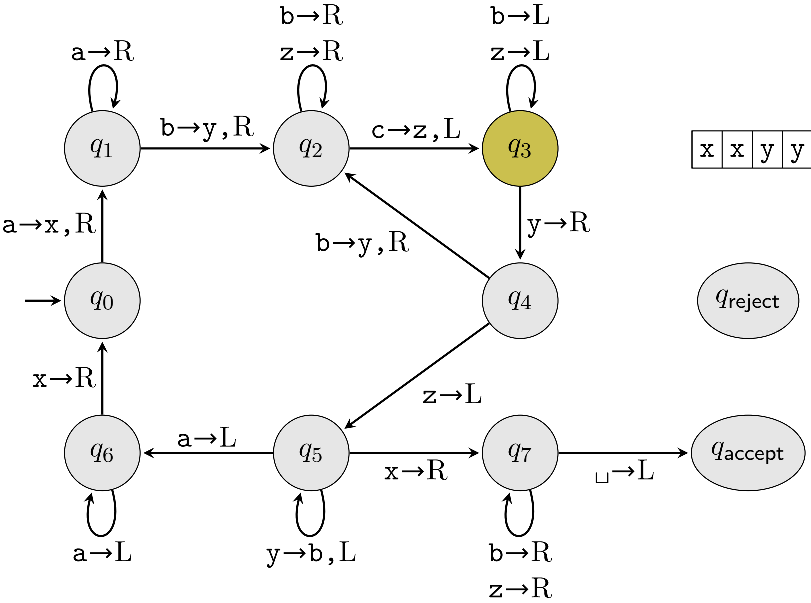
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	y	z	z	z	z	z	z	

Multiplication TM

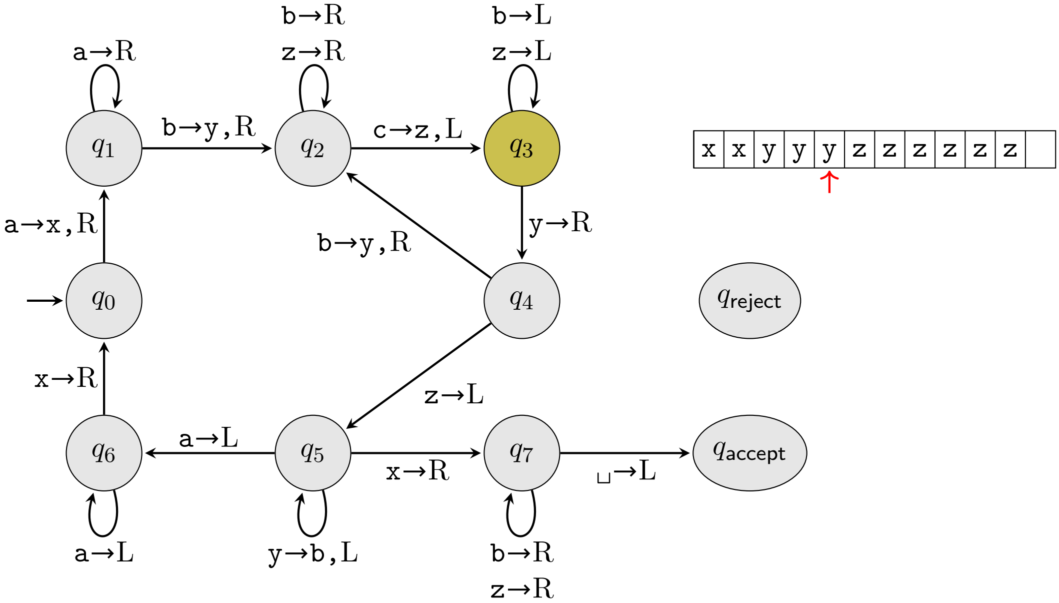
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	y	z	z	z	z	z	z	

Multiplication TM

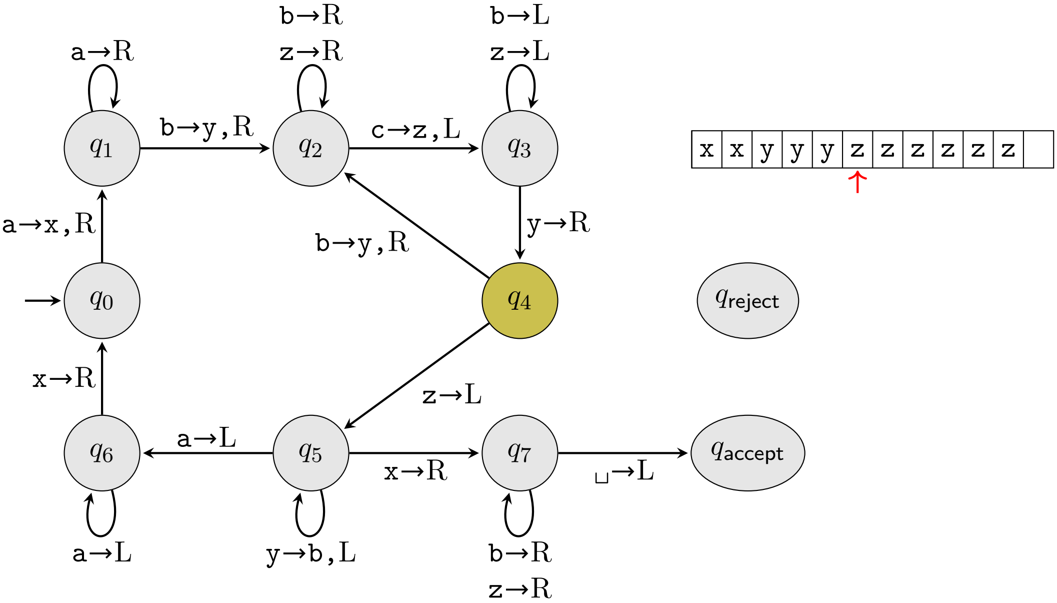
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \texttt{aabbbcccccc}$



x	x	y	y	y	z	z	z	z	z	z	

Multiplication TM

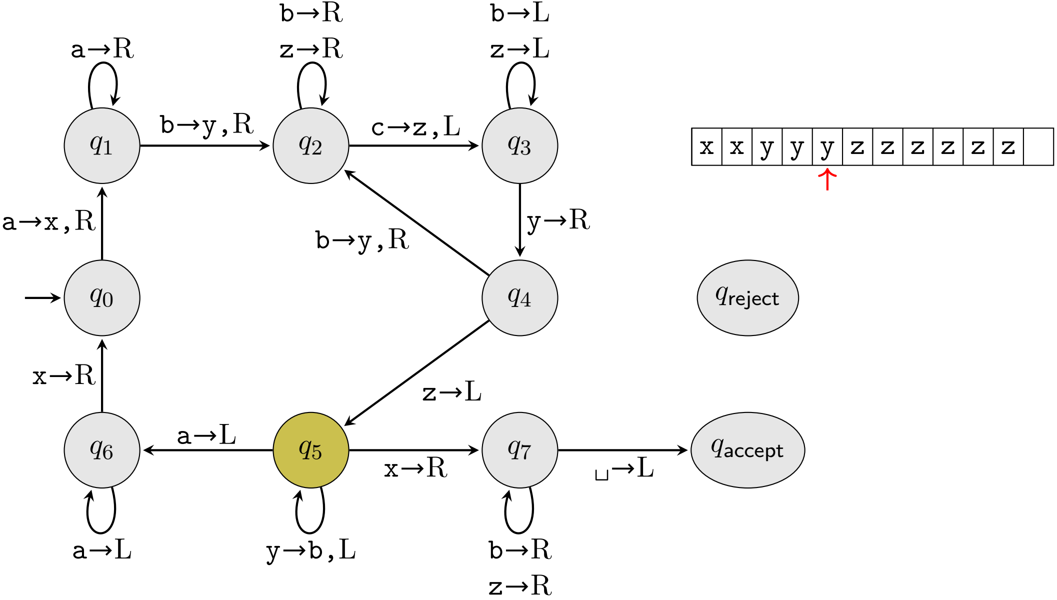
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \mathtt{aabbbcccccc}$



x	x	y	y	y	z	z	z	z	z	z	
					↑						

Multiplication TM

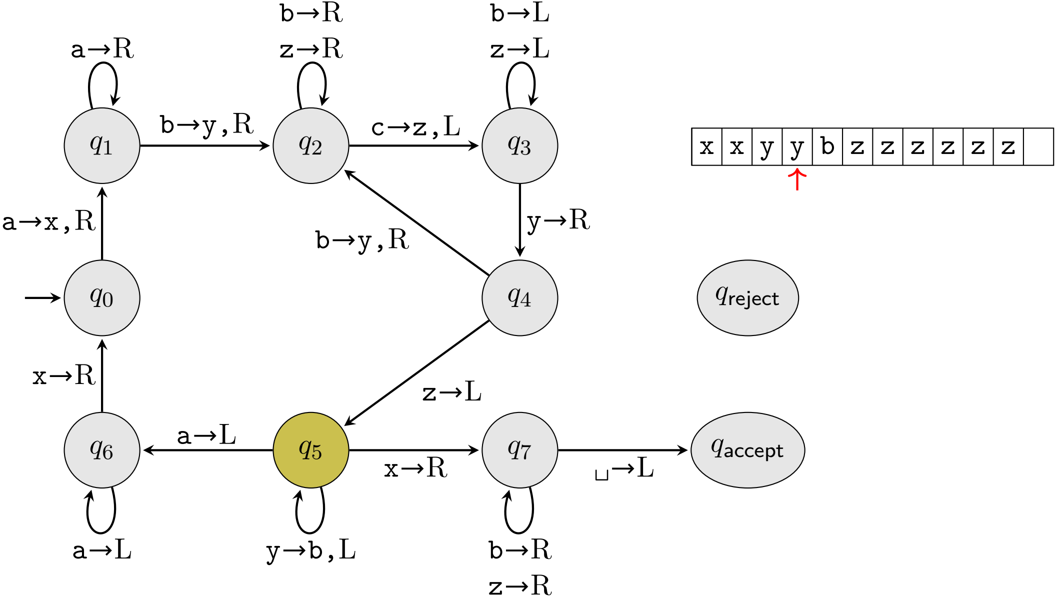
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	y	y	y	z	z	z	z	z	z	

Multiplication TM

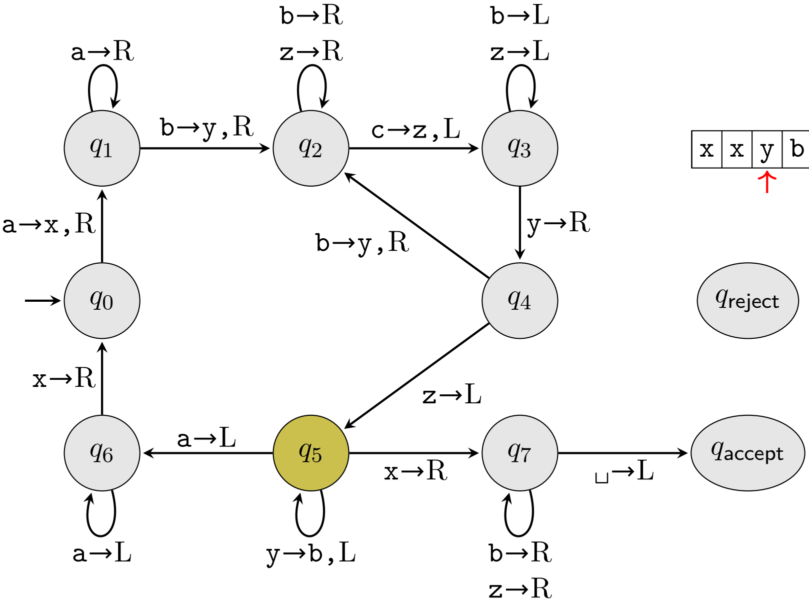
$B = \{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \texttt{aabbbcccccc}$



x	x	y	y	b	z	z	z	z	z	z	

Multiplication TM

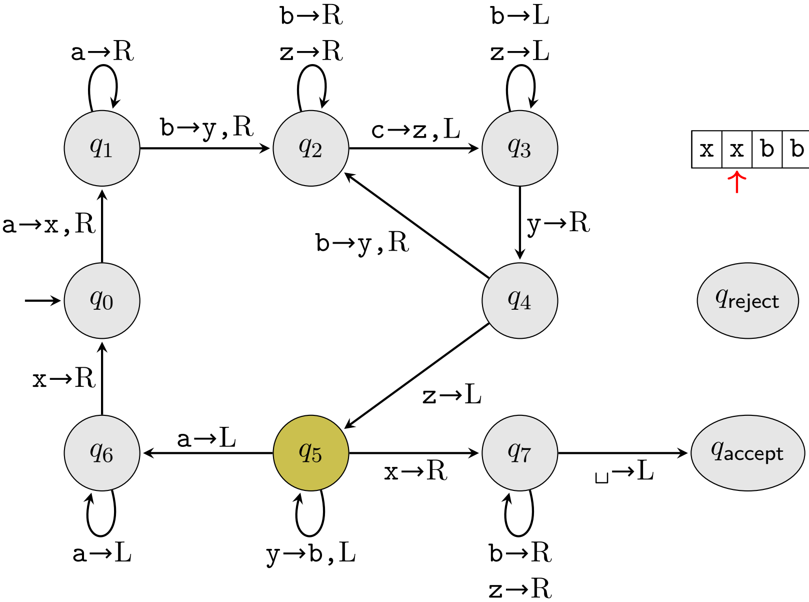
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \mathtt{aabbbcccccc}$



x	x	y	b	b	z	z	z	z	z	z	
		↑									

Multiplication TM

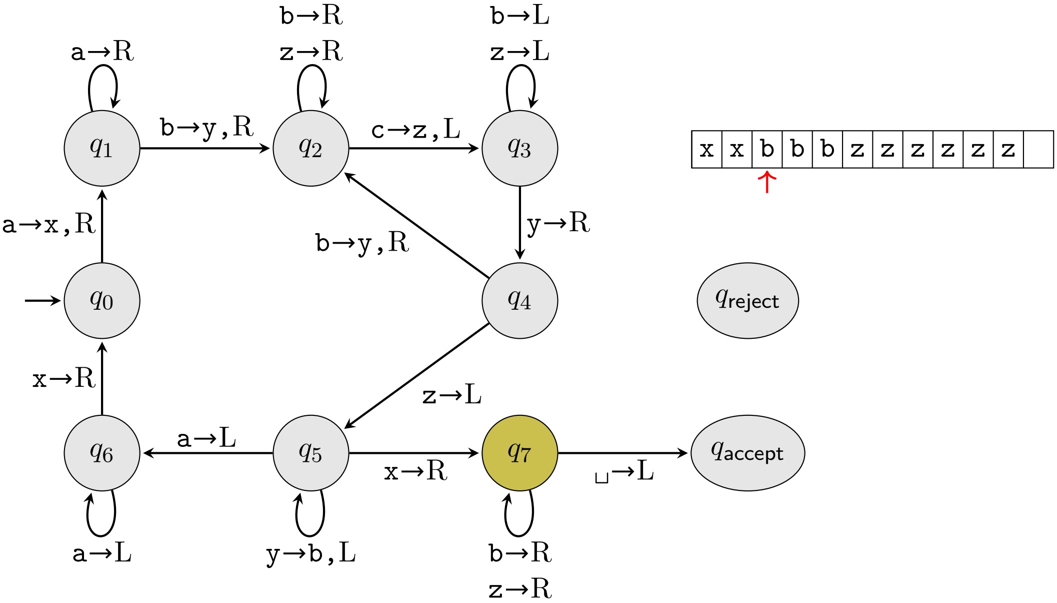
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	b	b	b	z	z	z	z	z	z	
	↑										

Multiplication TM

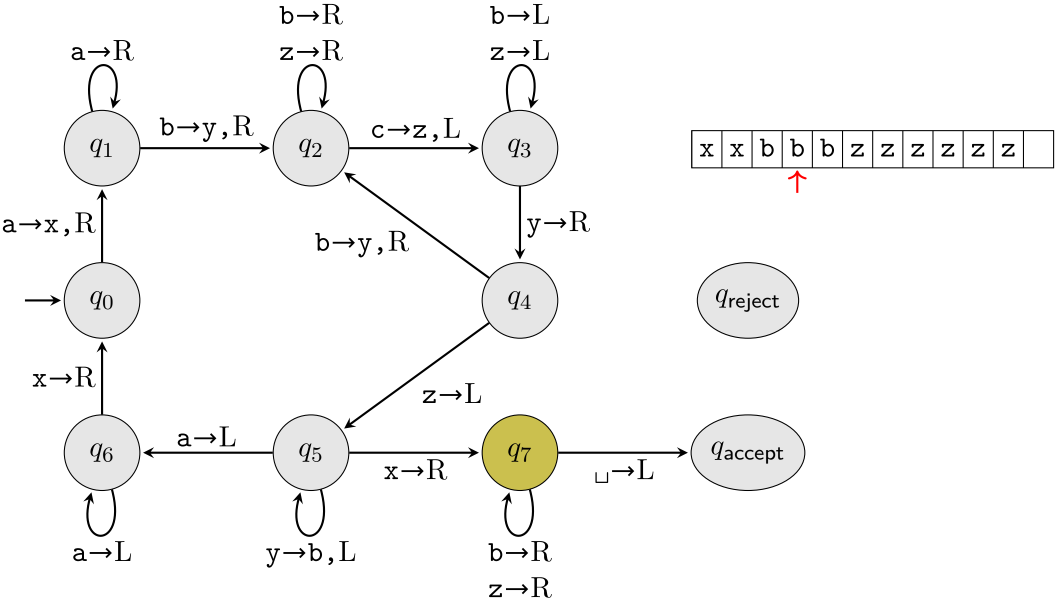
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	b	b	b	z	z	z	z	z	z	

Multiplication TM

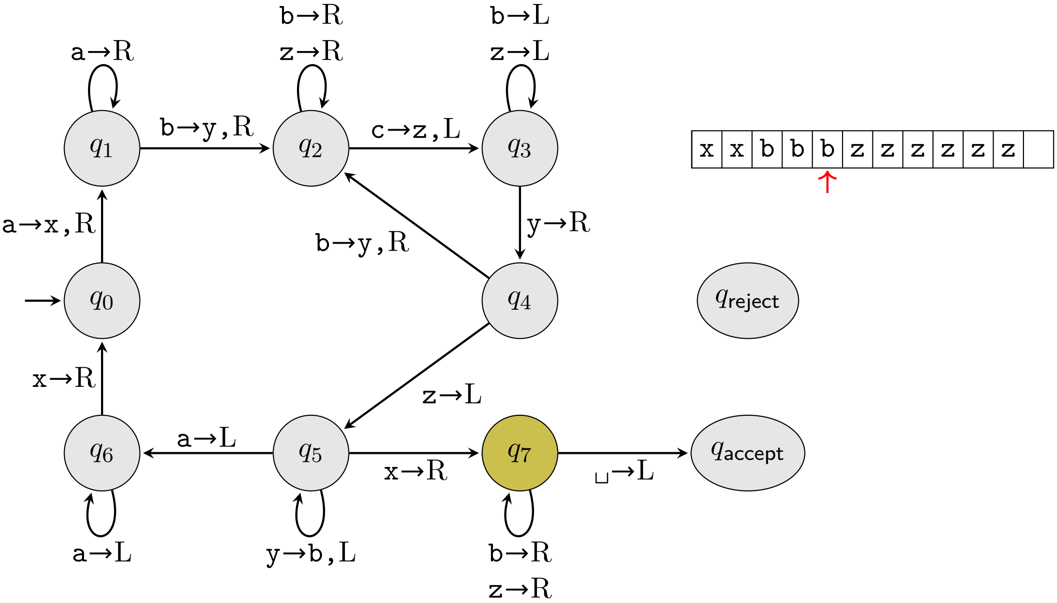
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $\quad w = \mathtt{aabbbcccccc}$



x	x	b	b	b	z	z	z	z	z	z	

Multiplication TM

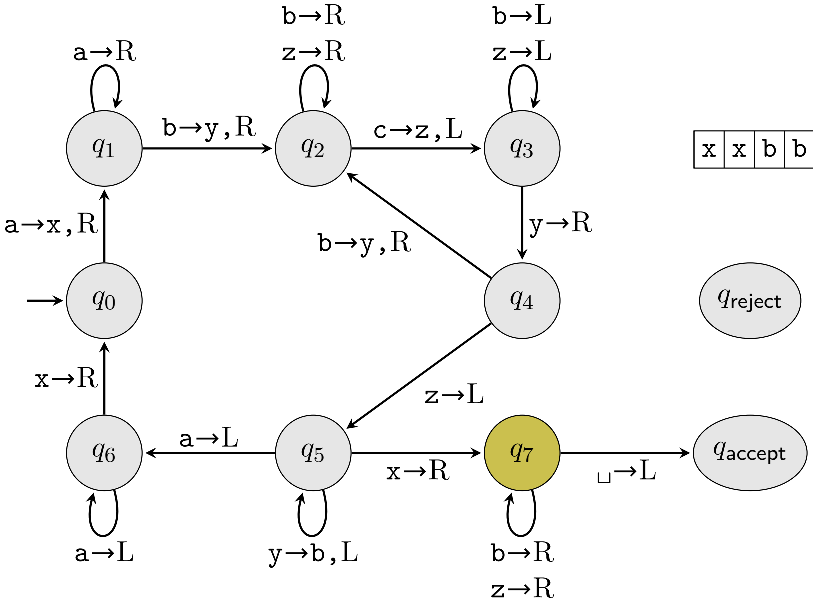
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	b	b	b	z	z	z	z	z	z	

Multiplication TM

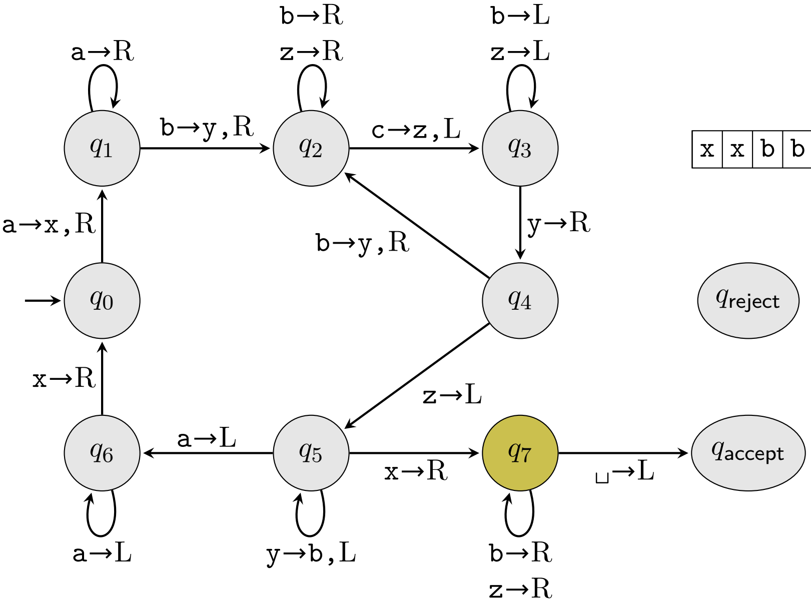
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i,j,k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	b	b	b	z	z	z	z	z	z	

Multiplication TM

$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbbcccccc}$



x	x	b	b	b	z	z	z	z	z	z	
						↑					

Multiplication TM

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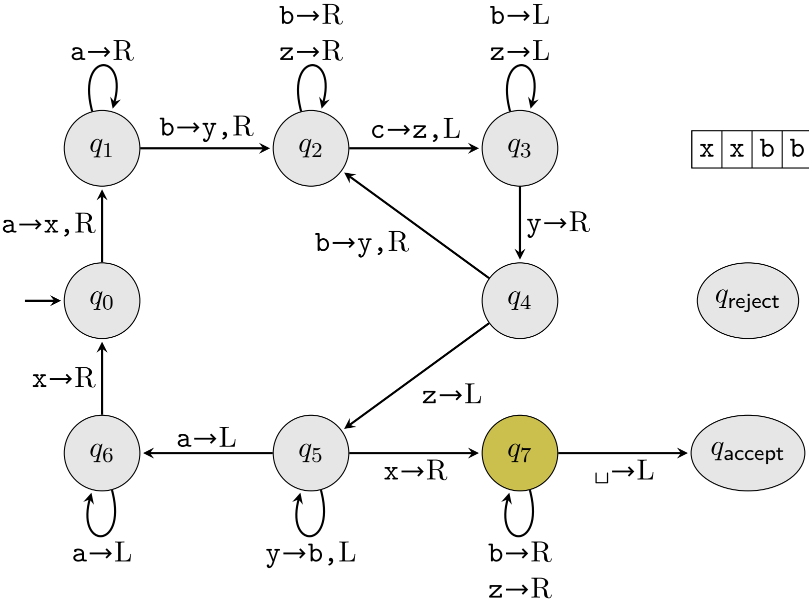

x	x	b	b	b	z	z	z	z	z	z	

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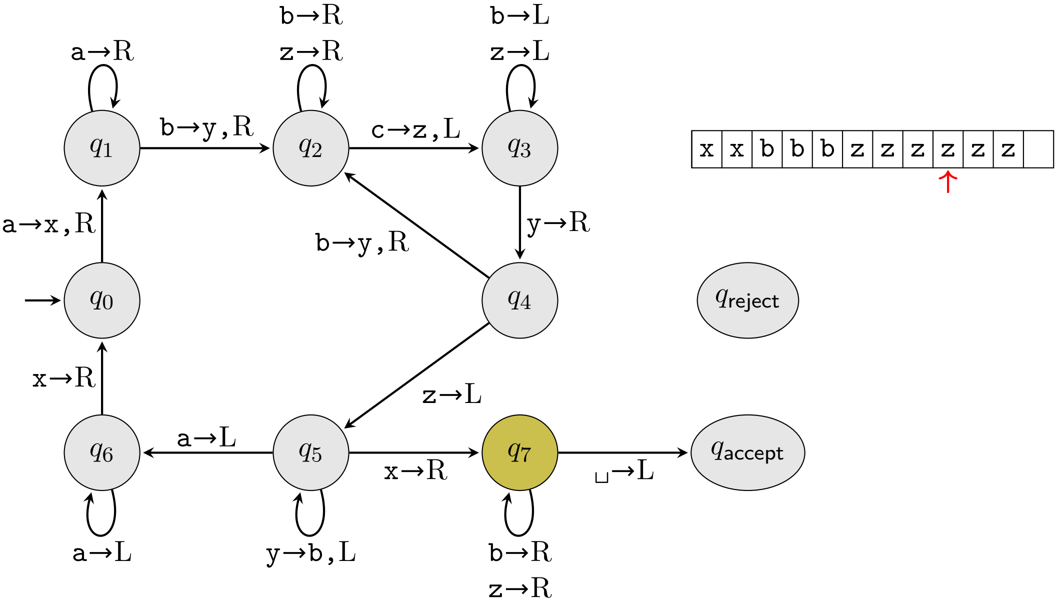

x	x	b	b	b	z	z	z	z	z	z	

Multiplication TM

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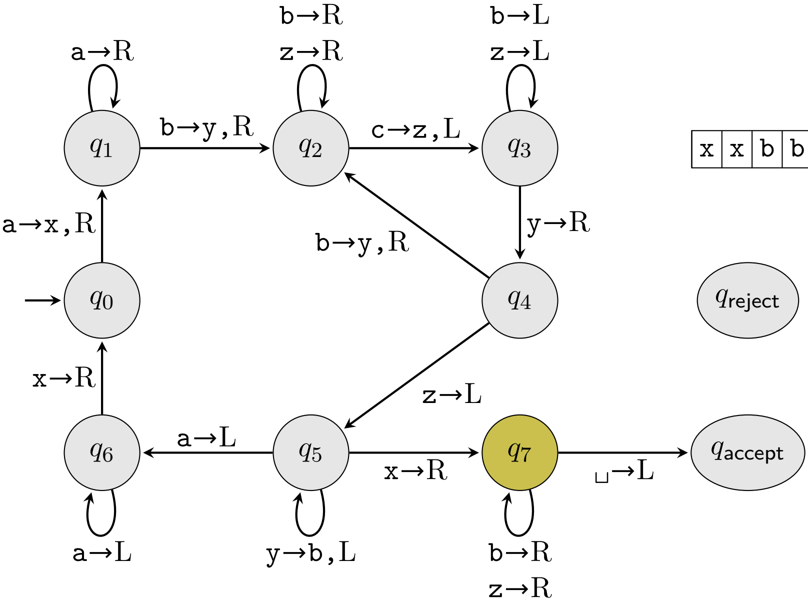

x	x	b	b	b	z	z	z	z	z	z	

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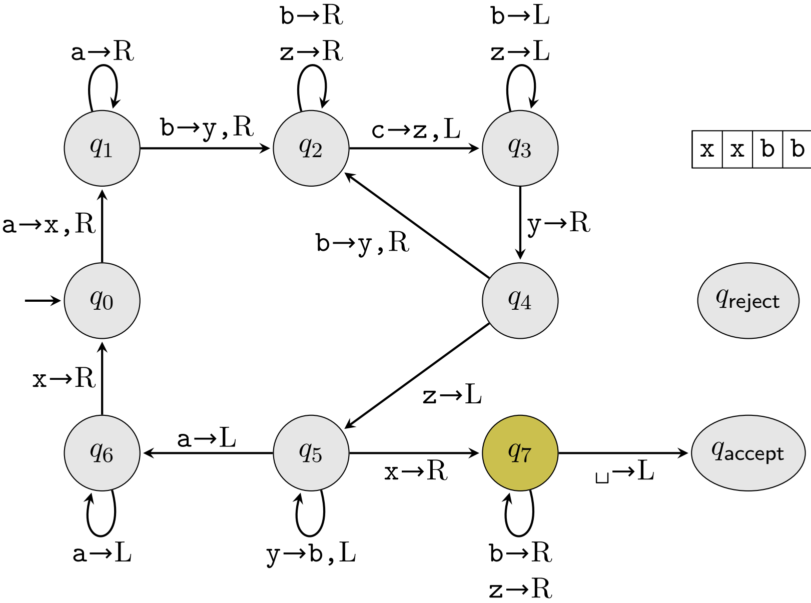

x	x	b	b	b	z	z	z	z	z	z	

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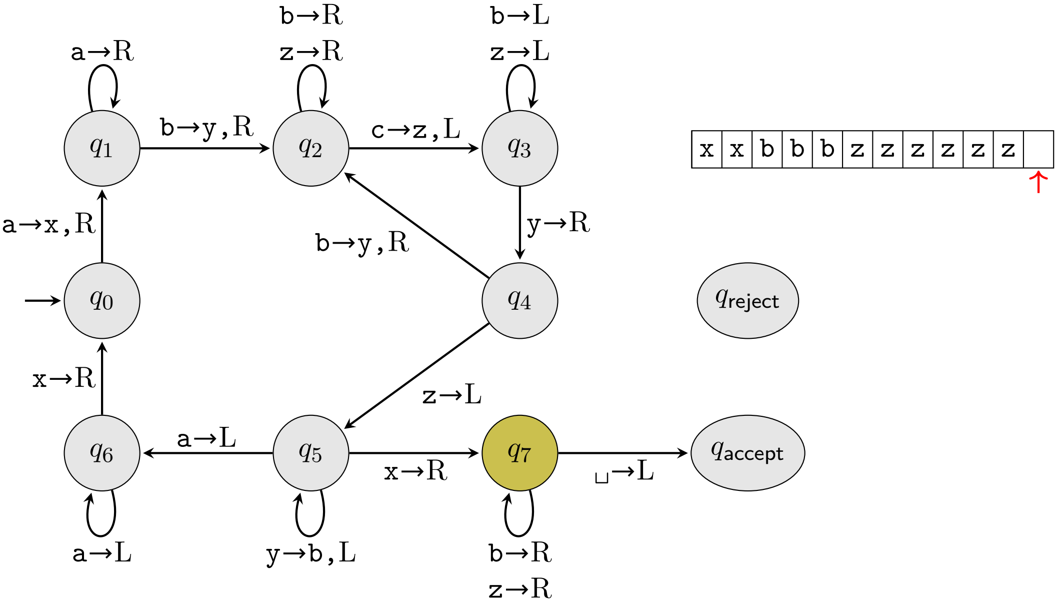

x	x	b	b	b	z	z	z	z	z	z	

Multiplication TM

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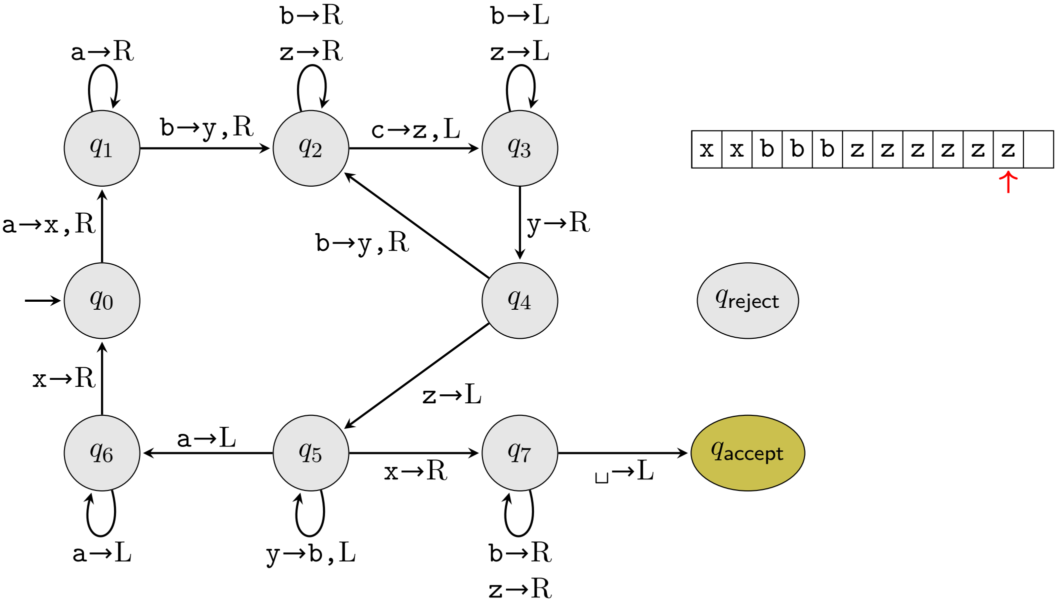
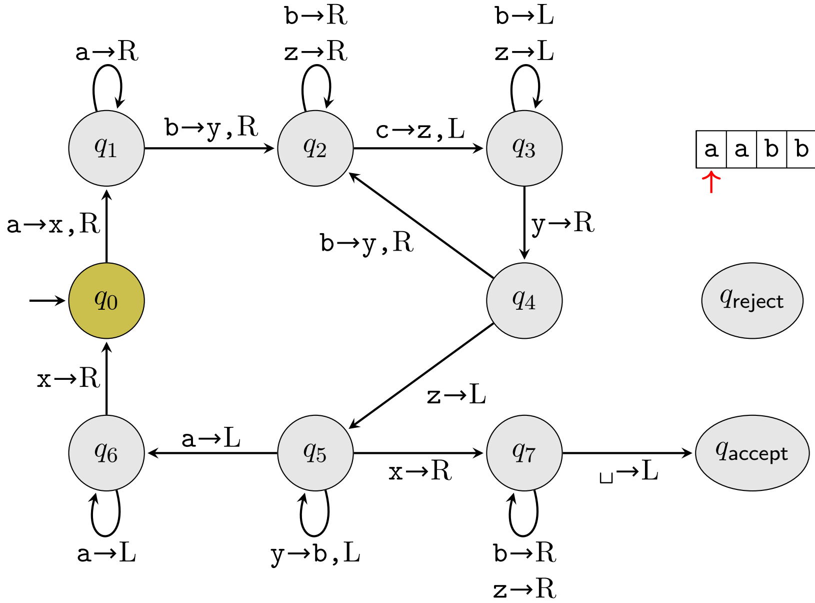


x	x	b	b	b	z	z	z	z	z	z	

Multiplication TM

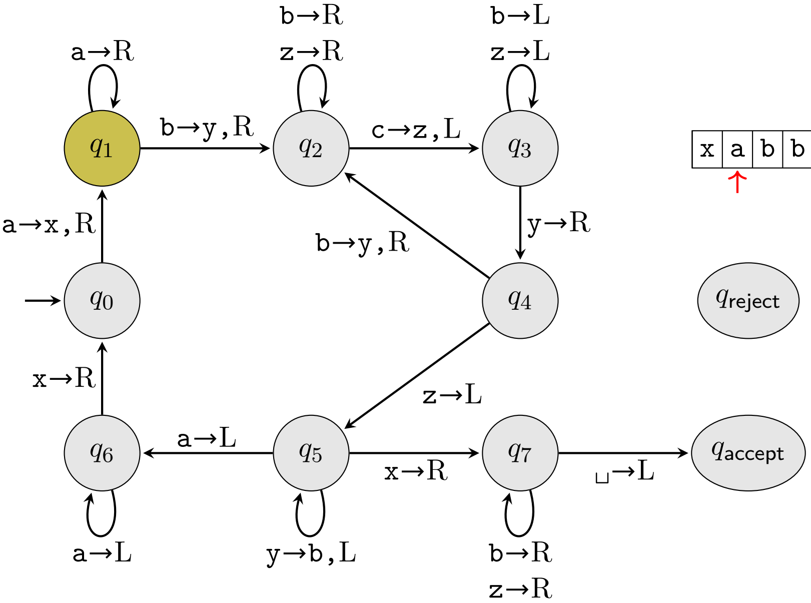
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbccc}$



a	a	b	b	c	c	c					

Multiplication TM

$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbccc}$



x	a	b	b	c	c	c					
	↑										

Multiplication TM

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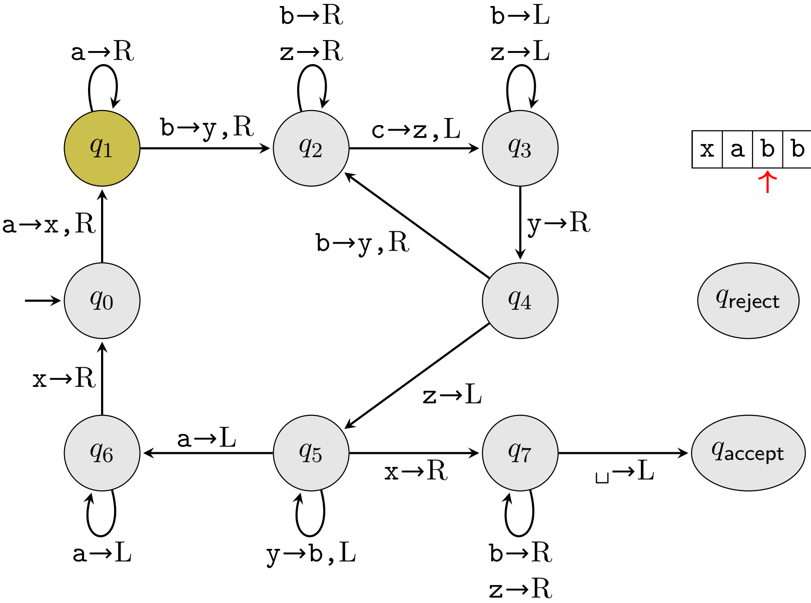

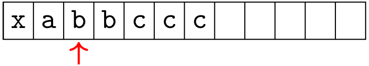

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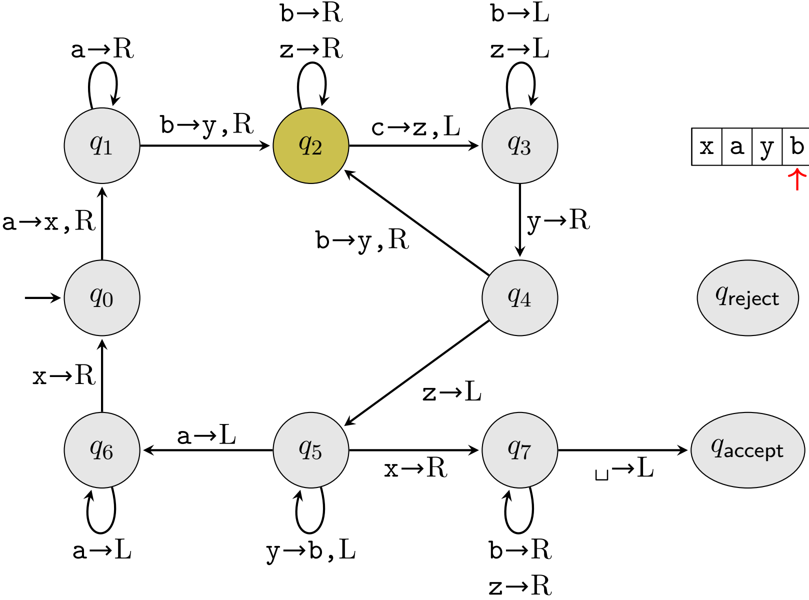
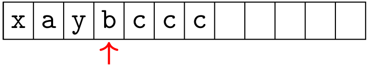
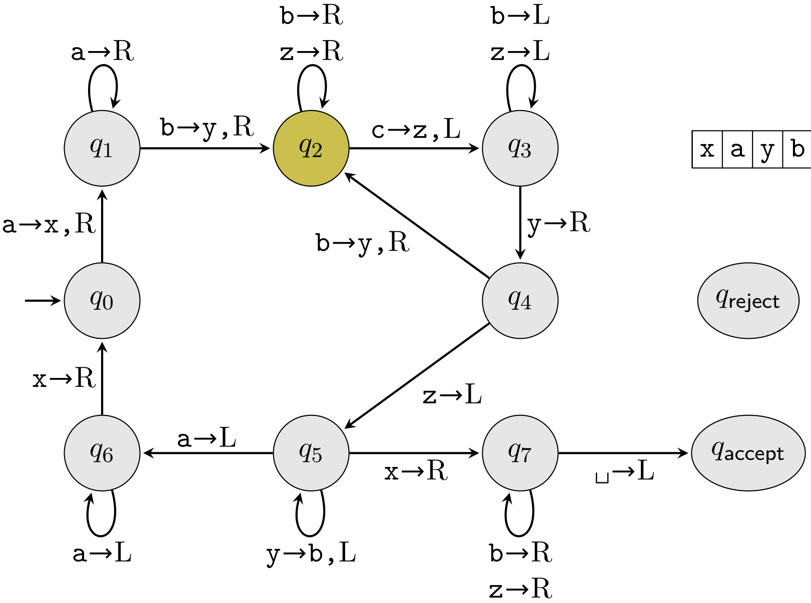




x	a	y	b	c	c	c					

Multiplication TM

$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbccc}$



x	a	y	b	c	c	c				
				↑						

Multiplication TM

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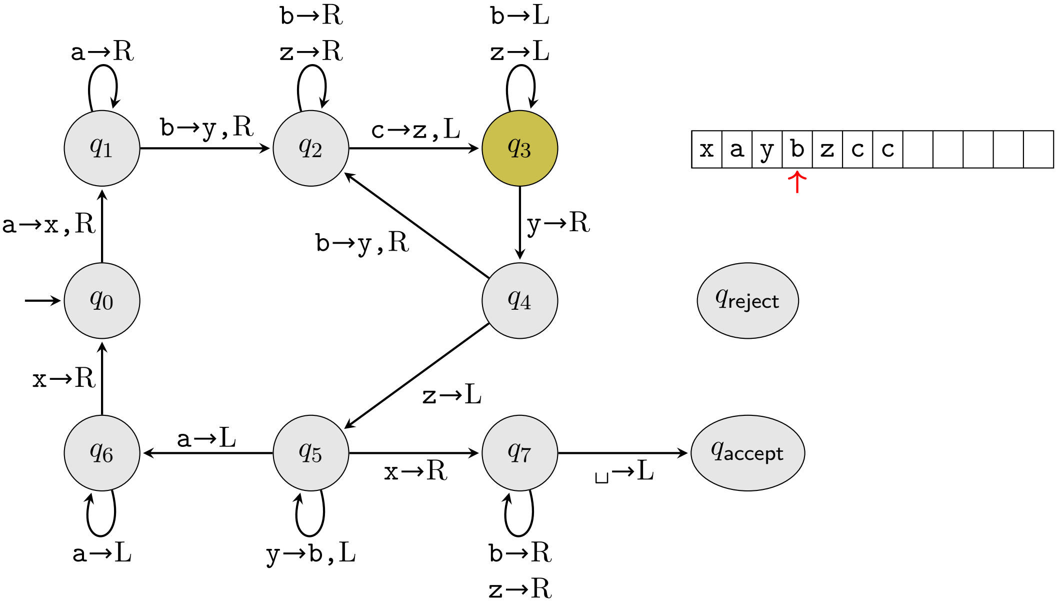

x	a	y	b	z	c	c					

Multiplication TM

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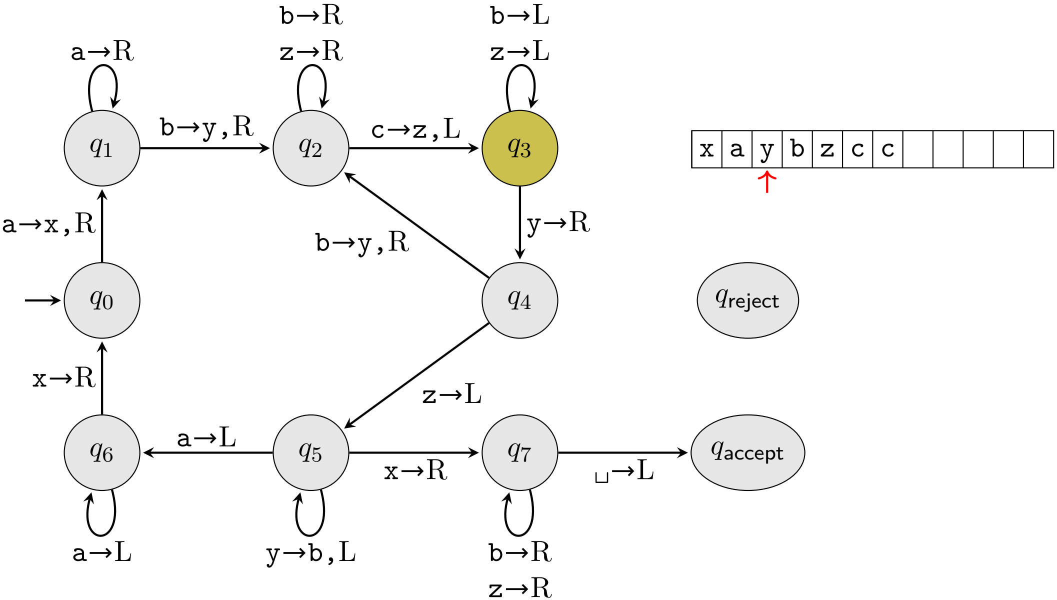
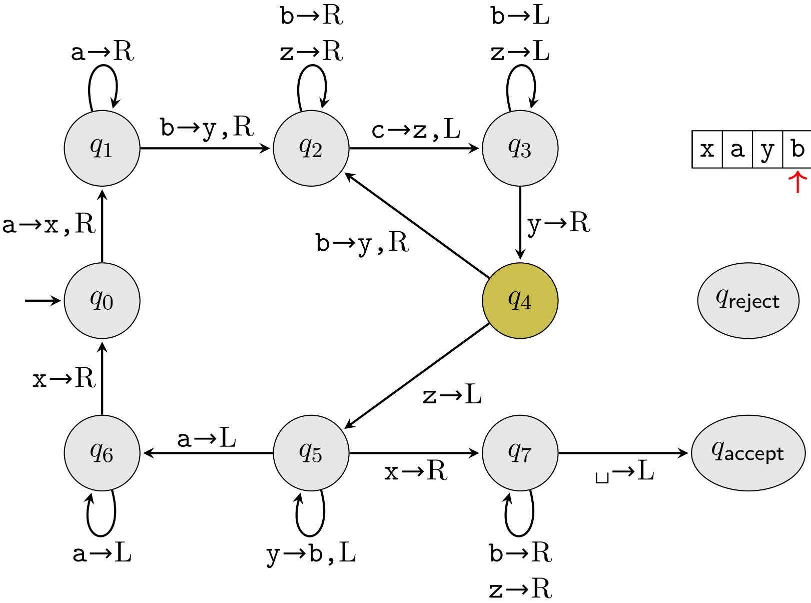
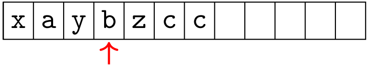


x	a	y	b	z	c	c					

Multiplication TM

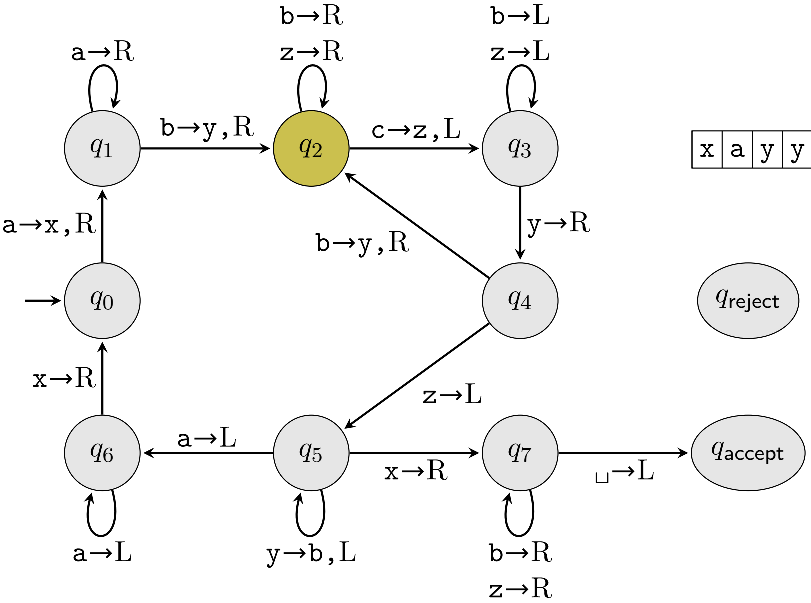
$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbccc}$





Multiplication TM

$B = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k > 0 \text{ and } i \cdot j = k\}$ $w = \mathtt{aabbccc}$



x	a	y	y	z	c	c					
				↑							

Multiplication TM

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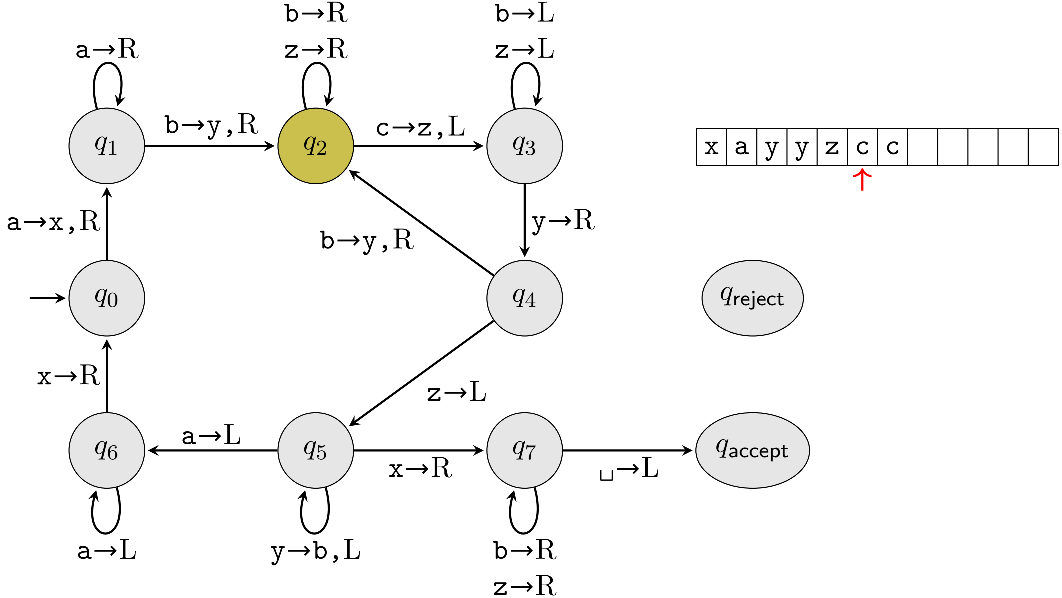

x	a	y	y	z	c	c					
					↑						

Multiplication TM

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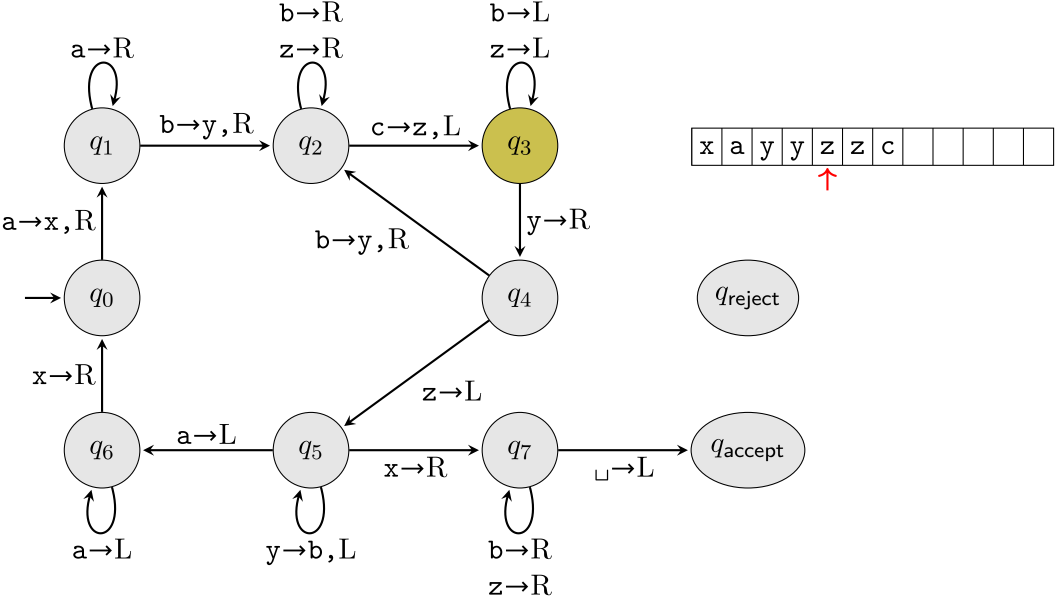

x	a	y	y	z	z	c					

Multiplication TM

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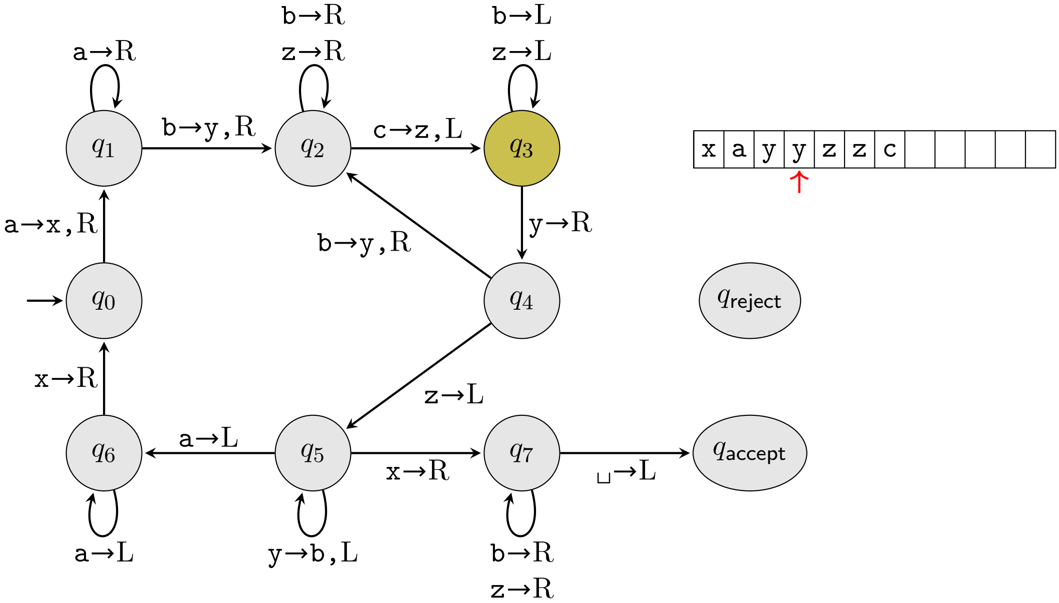

x	a	y	y	z	z	c					
			↑								

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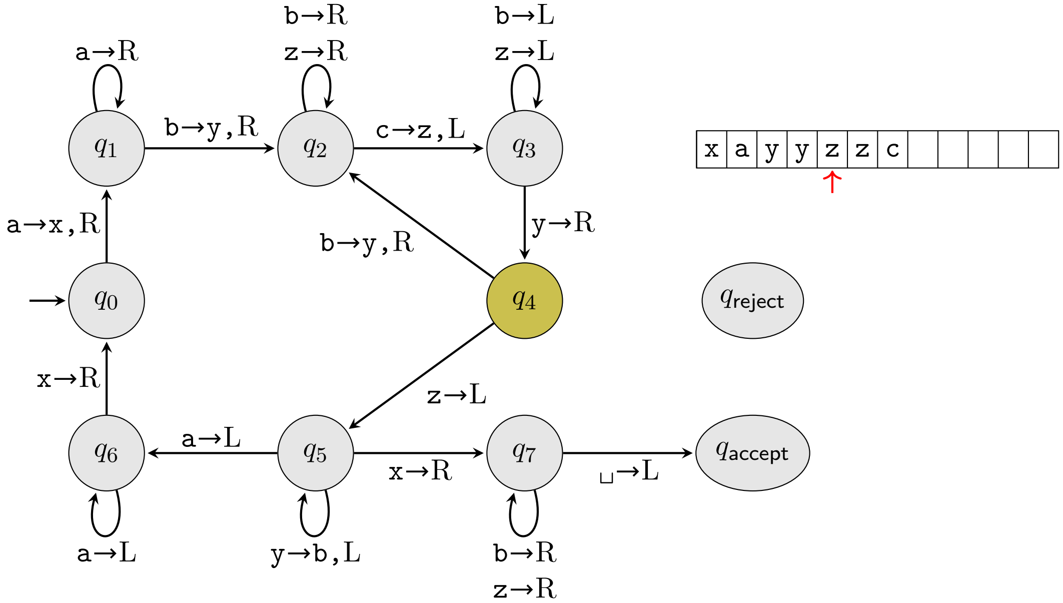

x	a	y	y	z	z	c					

Multiplication TM

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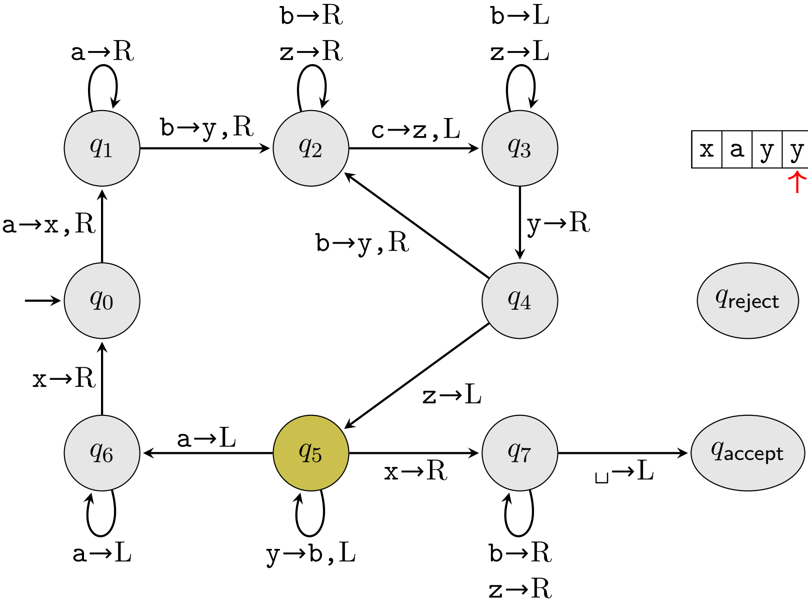

x	a	y	y	z	z	c					
			↑								

Multiplication TM

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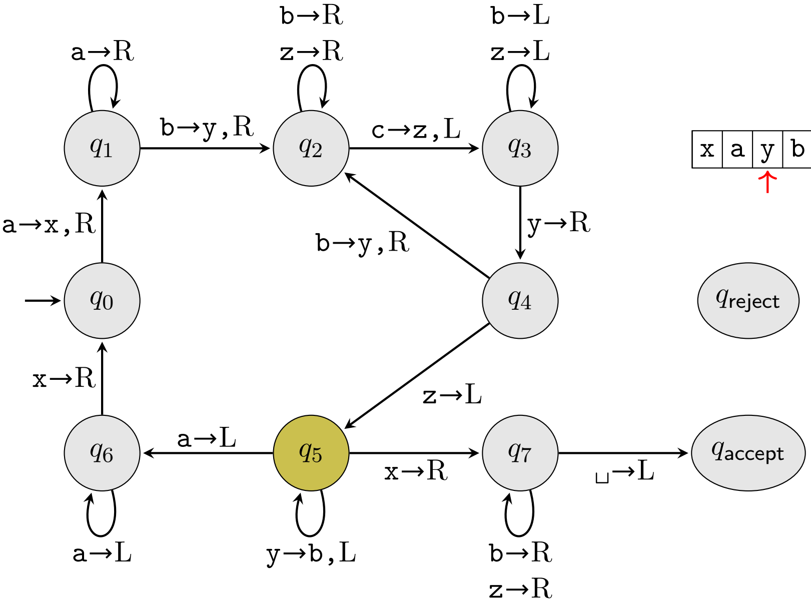

x	a	y	b	z	z	c					
		↑									

Multiplication TM

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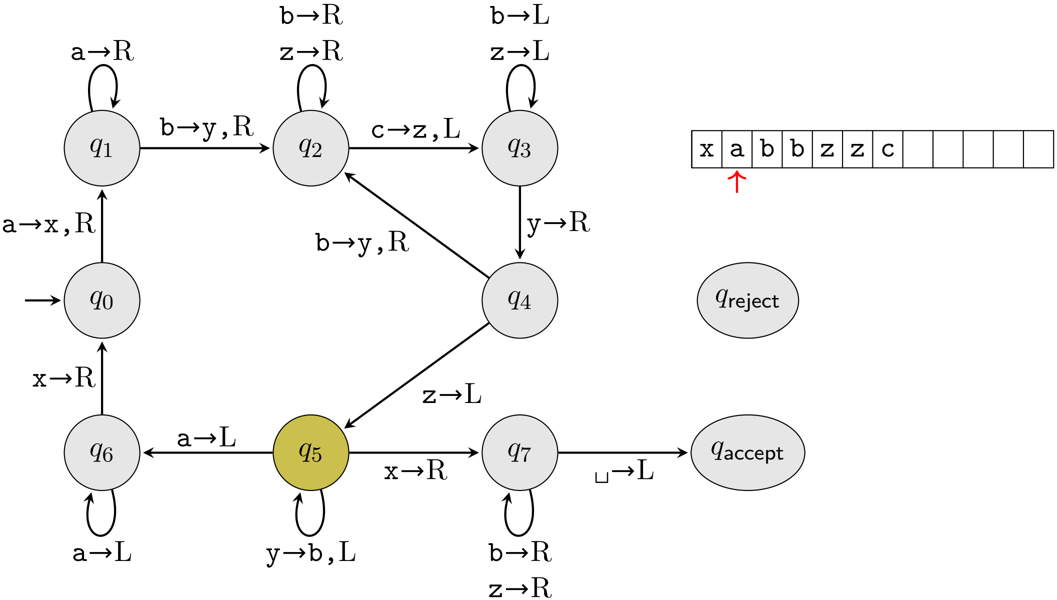

x	a	b	b	z	z	c					

Multiplication TM

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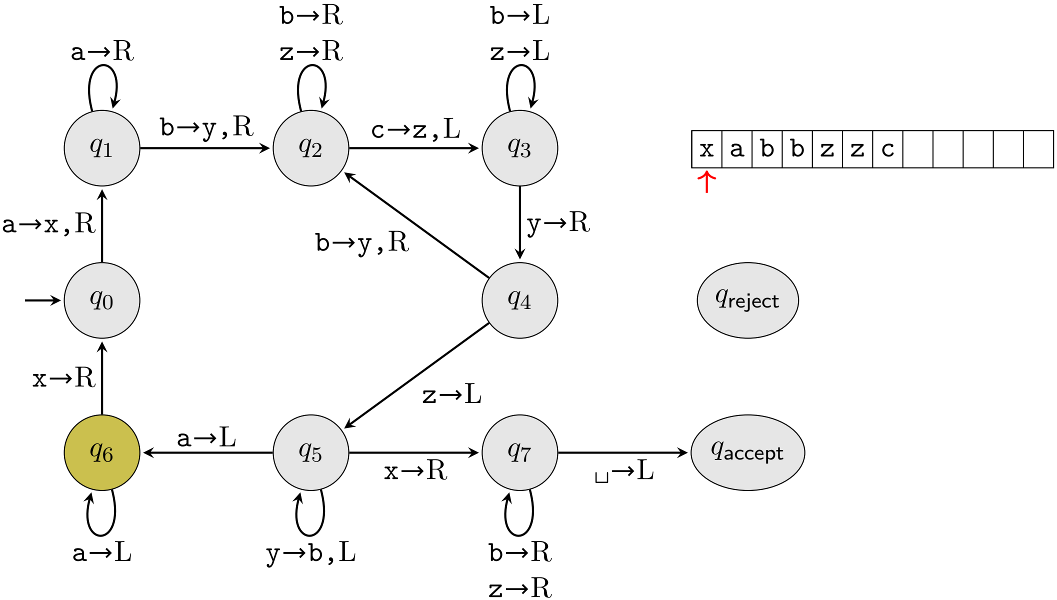

x	a	b	b	z	z	c					

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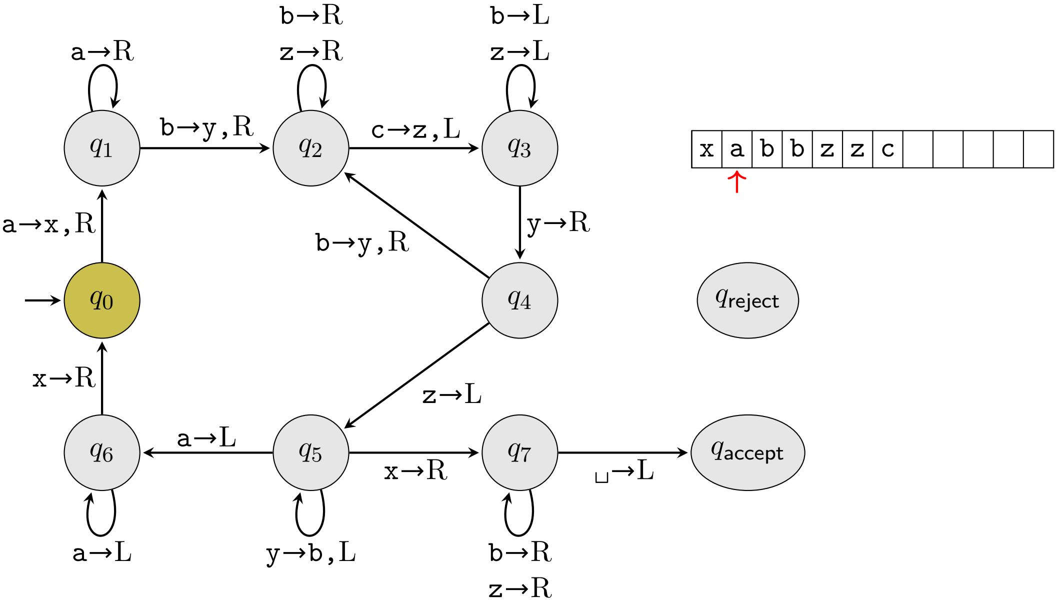

x	a	b	b	z	z	c				

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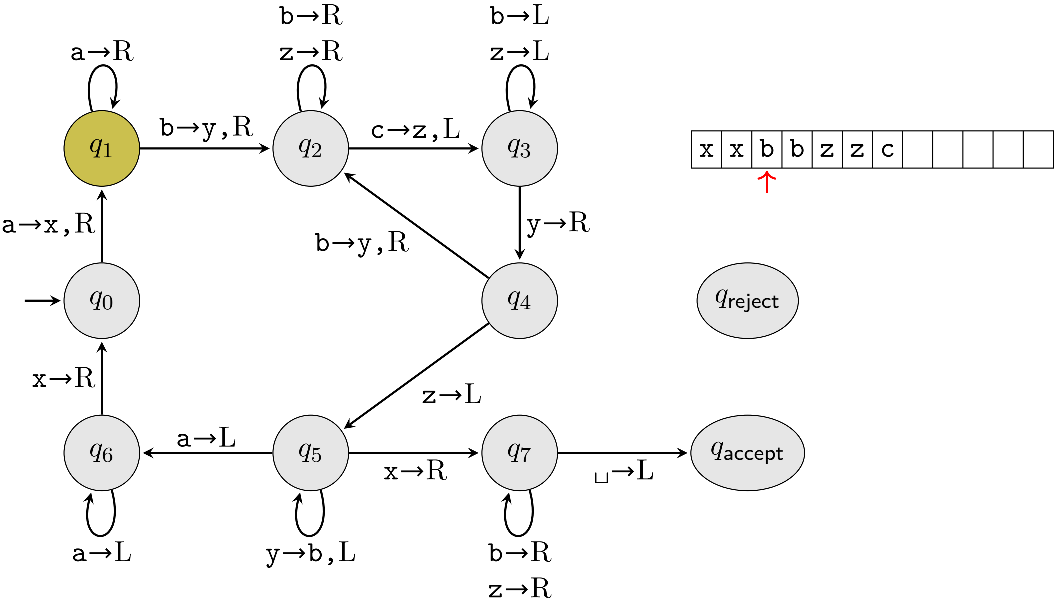

x	x	b	b	z	z	c					

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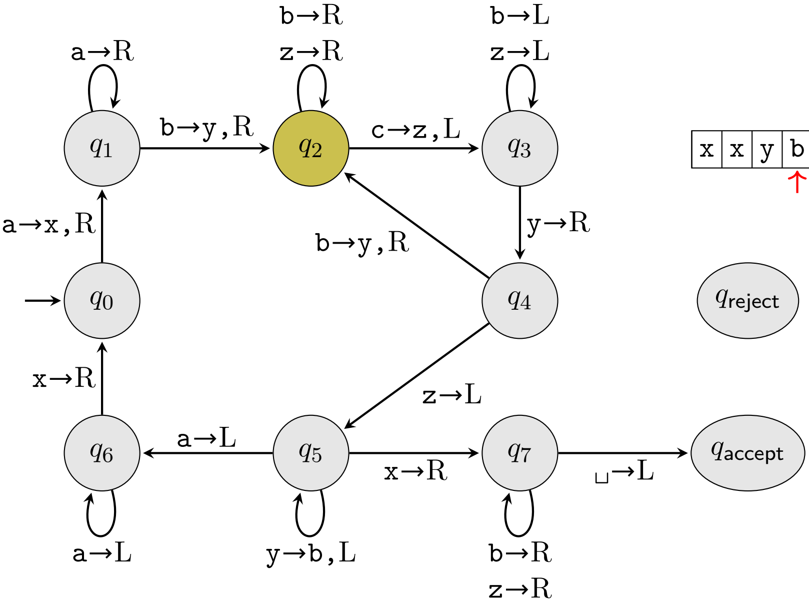

x	x	y	b	z	z	c					

Multiplication TM

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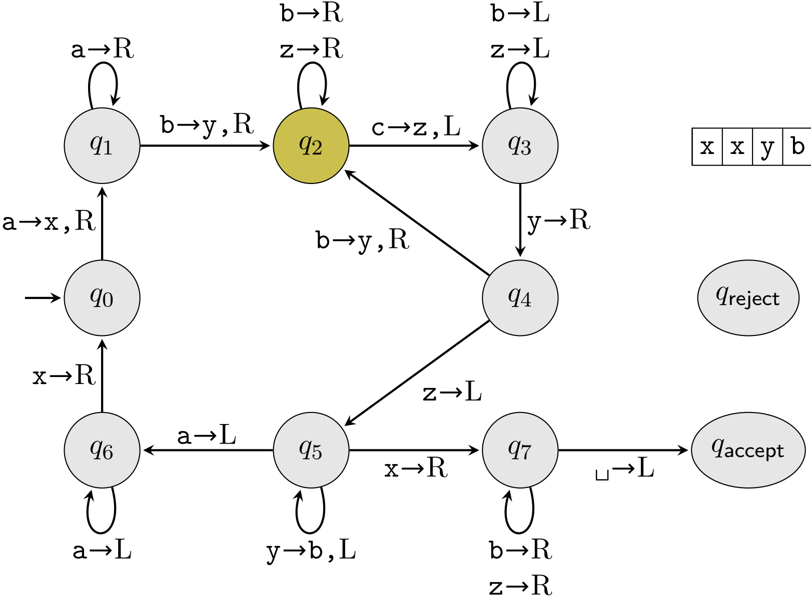

x	x	y	b	z	z	c					

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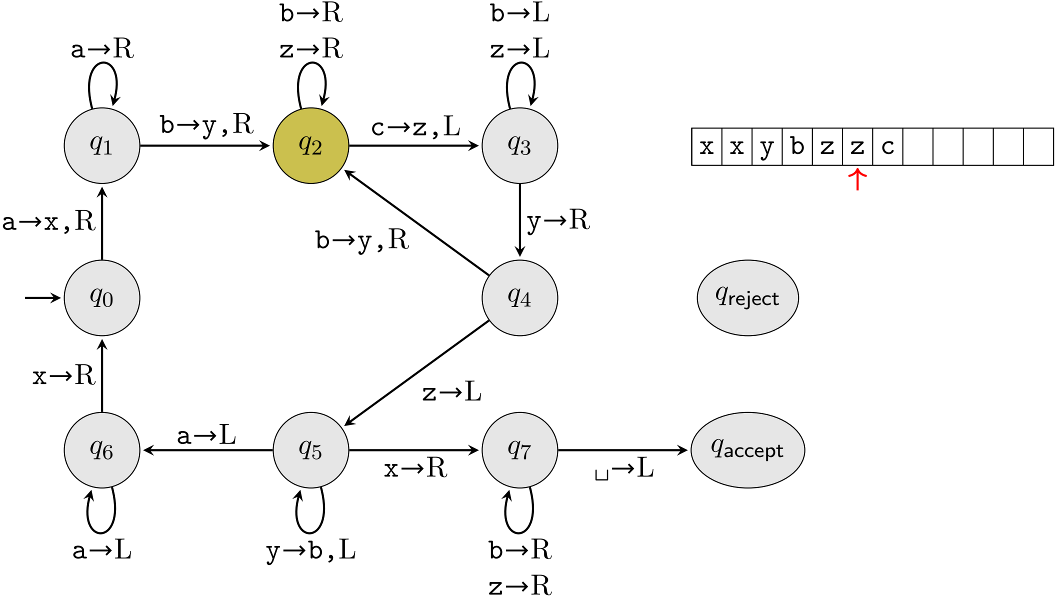

x	x	y	b	z	z	c					

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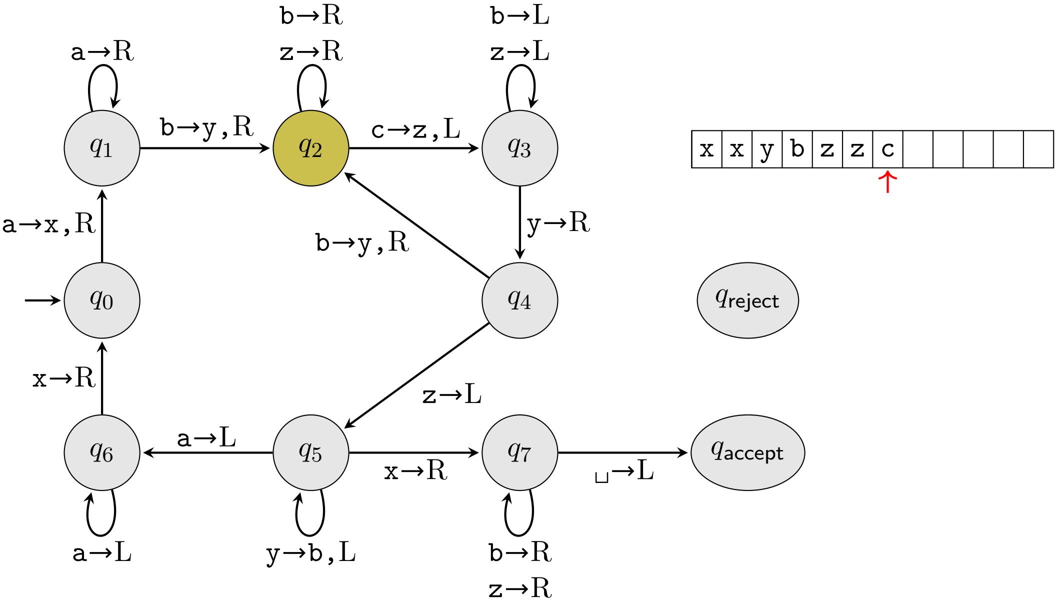

x	x	y	b	z	z	c					

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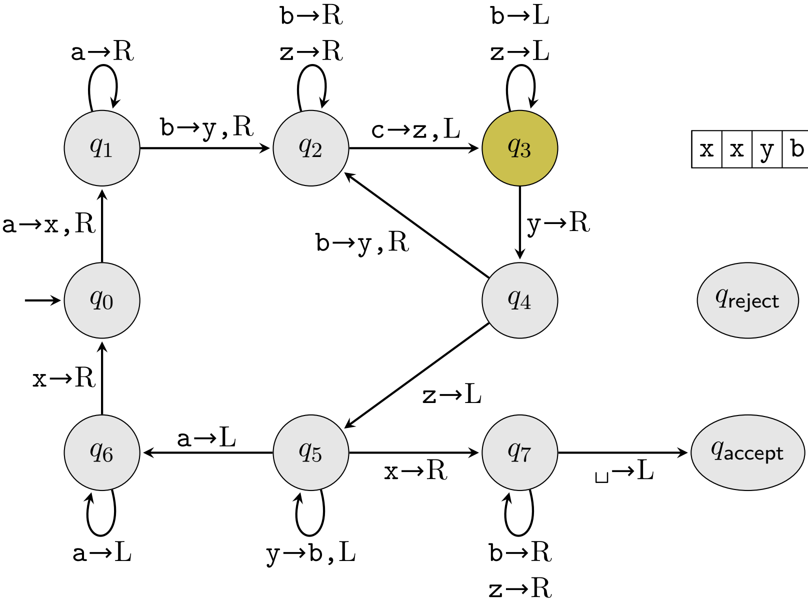

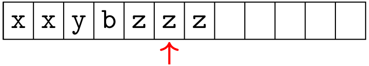

x	x	y	b	z	z	z				

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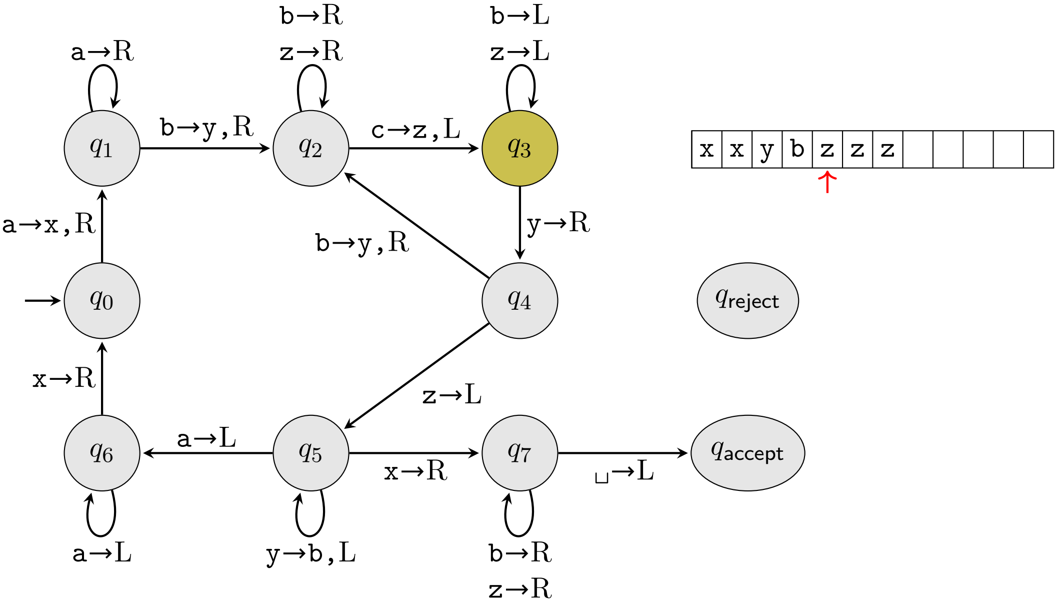

x	x	y	b	z	z	z					
				↑							

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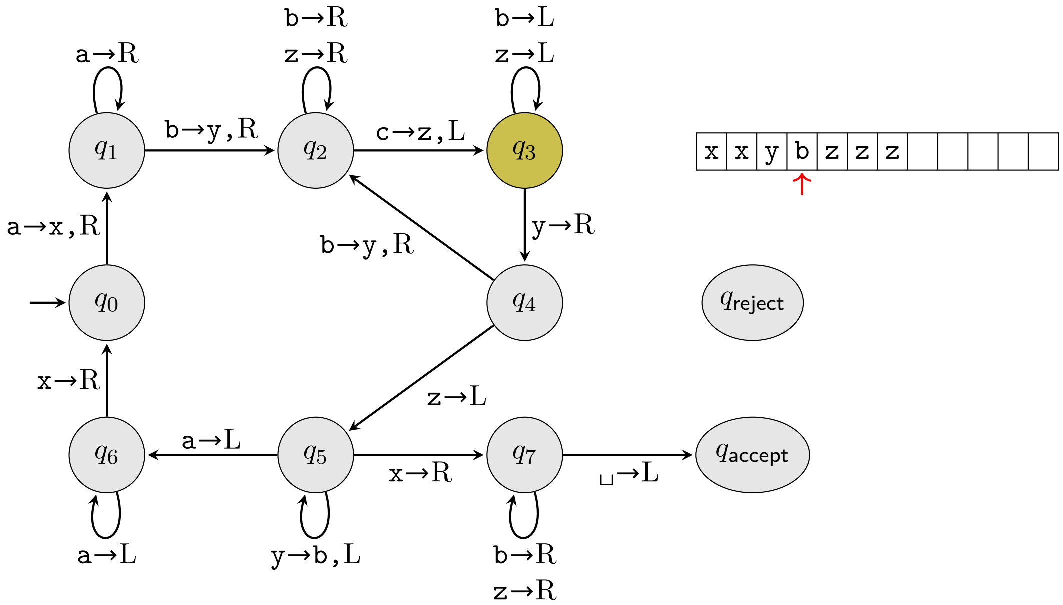

x	x	y	b	z	z	z					

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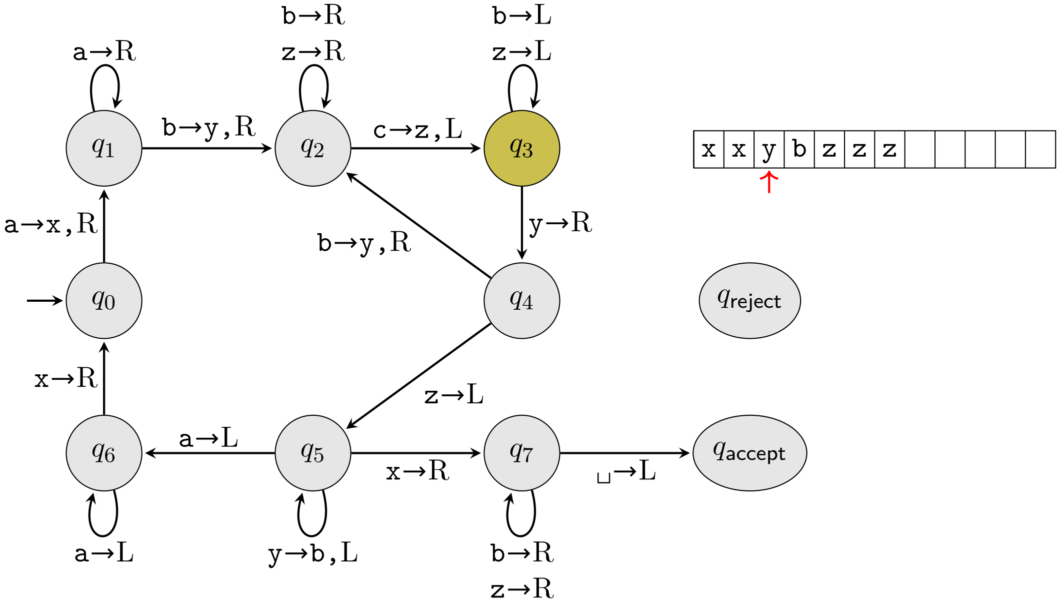

x	x	y	b	z	z	z					

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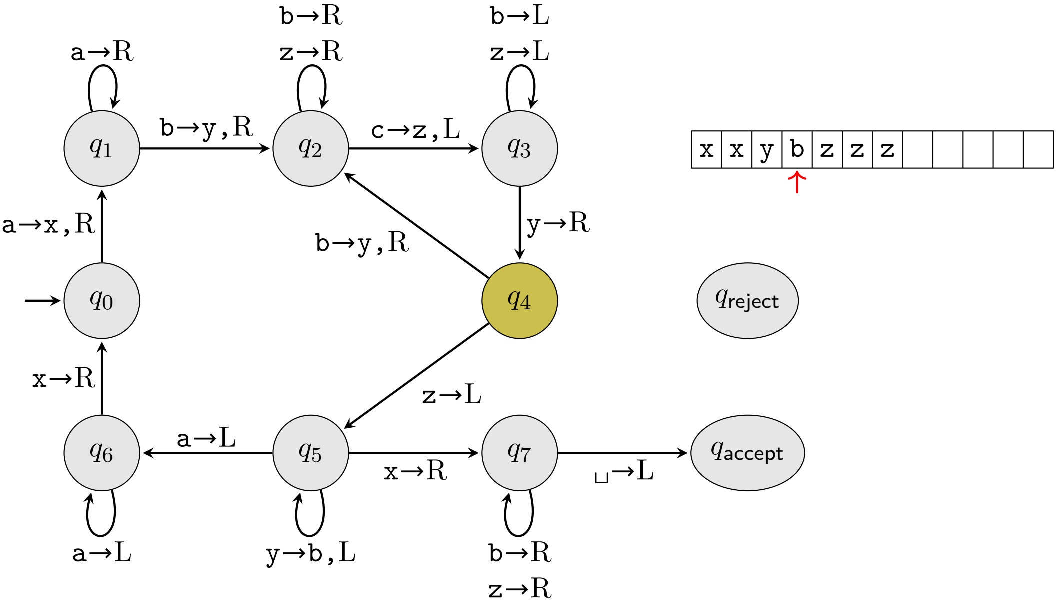

x	x	y	b	z	z	z					

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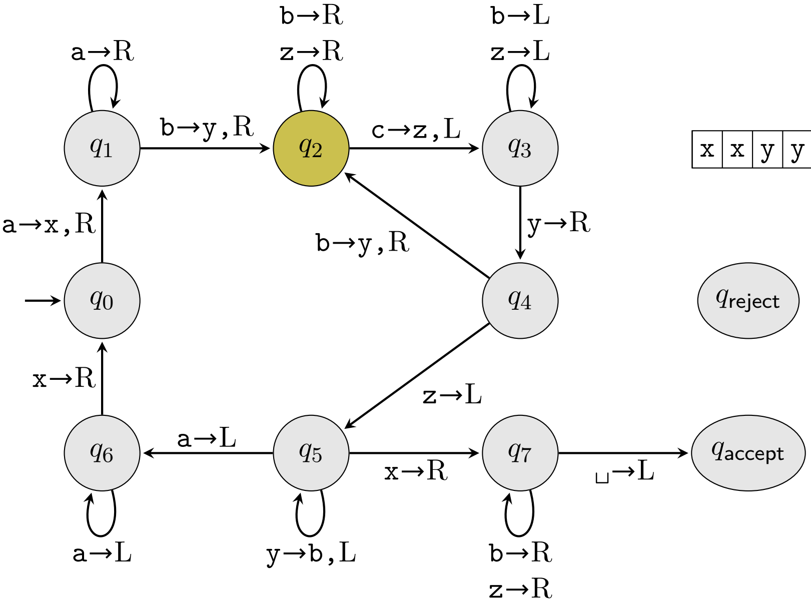

x	x	y	y	z	z	z					
				↑							

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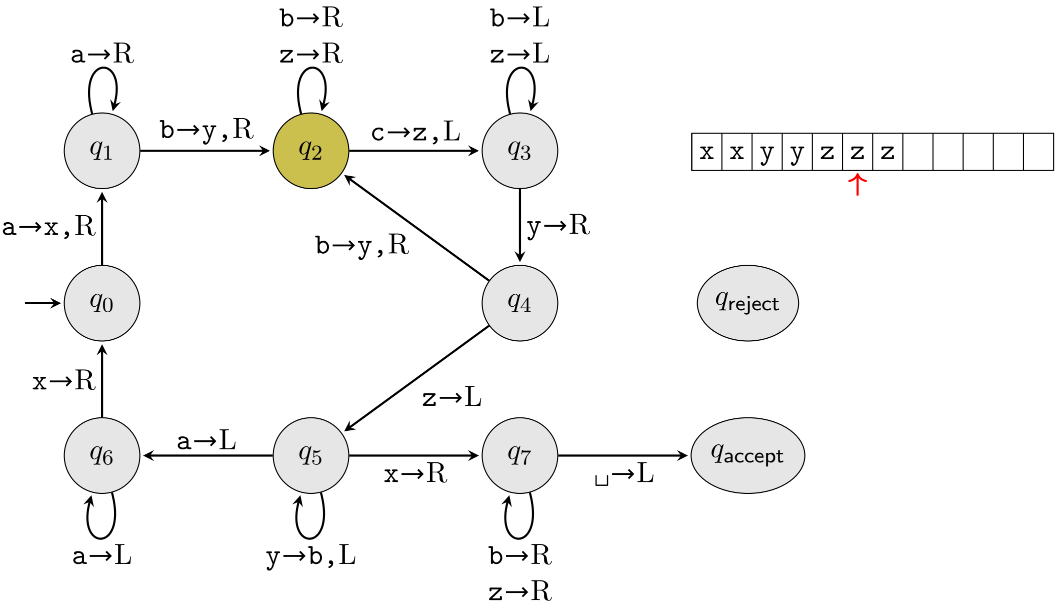

x	x	y	y	z	z	z				
					↑					

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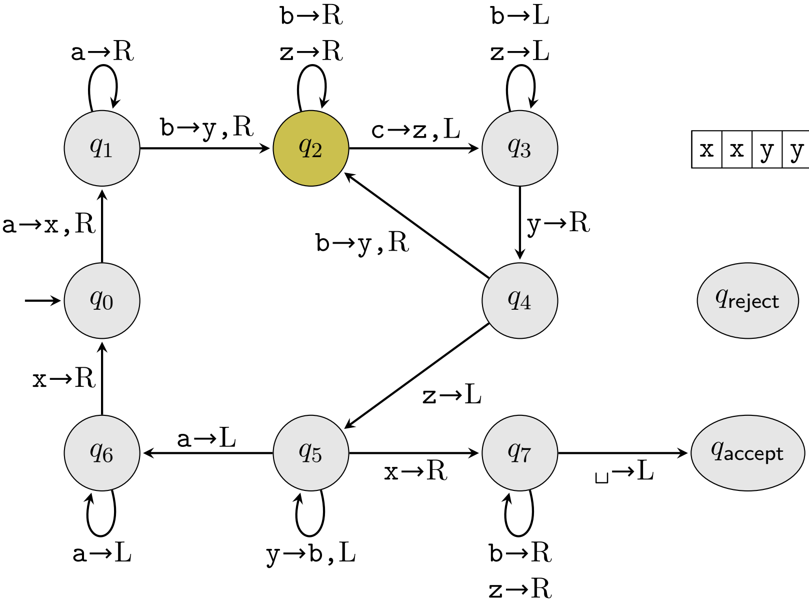

x	x	y	y	z	z	z					

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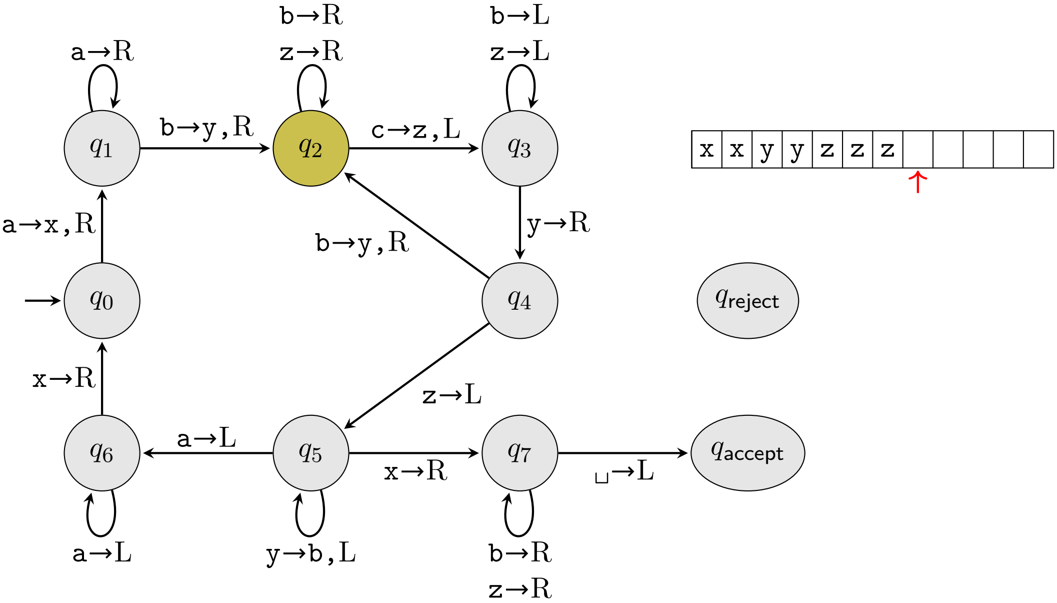

x	x	y	y	z	z	z					
							↑				

Multiplication TM

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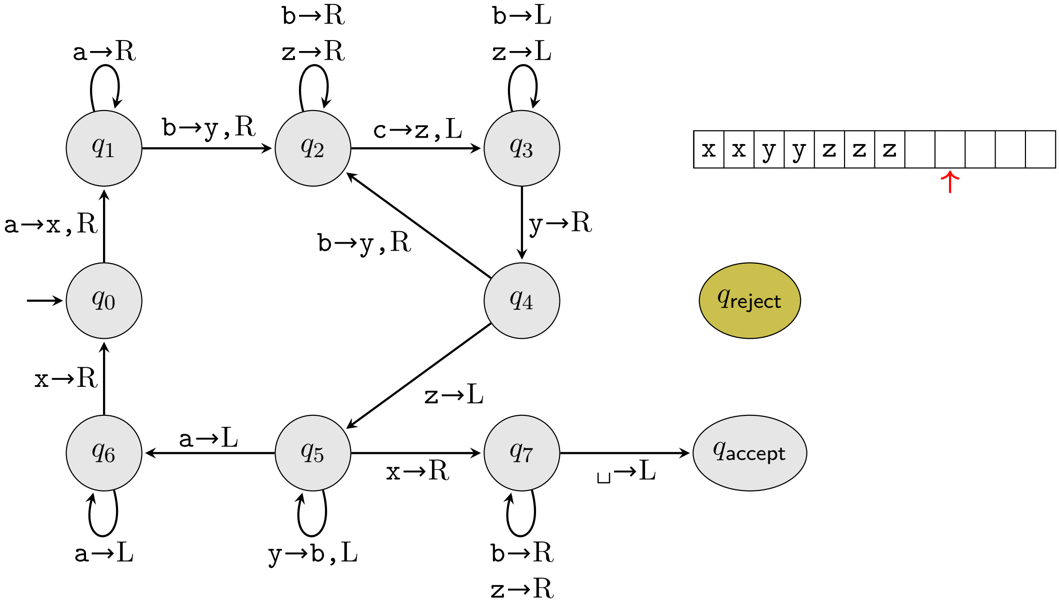


x	x	y	y	z	z	z					

Higher-level descriptions of TMs

Only the simplest TMs are reasonable to actually construct diagrams for

Instead, we want to give a higher-level description of a TM

For now, we'll stick with "implementation-level" descriptions, but soon we'll describe high-level algorithms

Implementation-level descriptions of TMs

Give a sequence of steps that a TM would follow

Examples

Implementation-level descriptions of TMs

Give a sequence of steps that a TM would follow

Examples

- "Check that the input matches some regular expression"
  This is easy because we could convert the regex to a DFA and encode the DFA in the TM. The TM would make a single pass over the input (without changing anything) and check if the DFA would accept

Implementation-level descriptions of TMs

Give a sequence of steps that a TM would follow

Examples

- "Check that the input matches some regular expression"
  This is easy because we could convert the regex to a DFA and encode the DFA in the TM. The TM would make a single pass over the input (without changing anything) and check if the DFA would accept
- "Mark a cell"
  For each symbol $t \in \Gamma$ we need to mark, add another symbol, say $\dot{t}$ to Γ

Implementation-level descriptions of TMs

Give a sequence of steps that a TM would follow

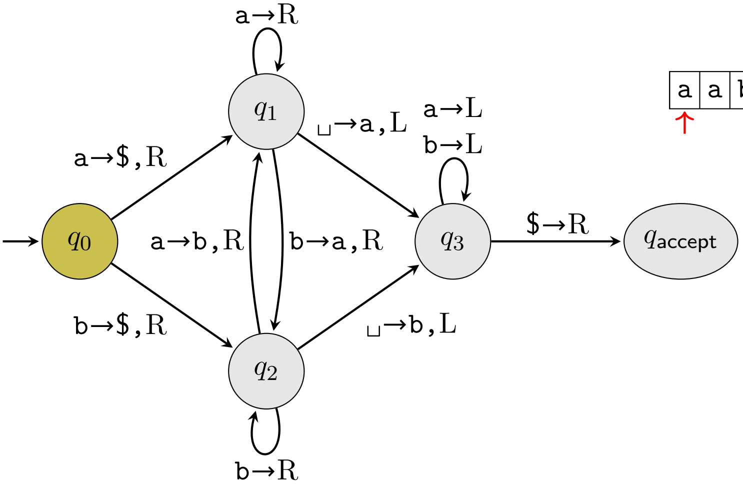
Examples

- "Check that the input matches some regular expression"
  This is easy because we could convert the regex to a DFA and encode the DFA in the TM. The TM would make a single pass over the input (without changing anything) and check if the DFA would accept
- "Mark a cell"
  For each symbol $t \in \Gamma$ we need to mark, add another symbol, say $\dot{t}$ to Γ
- "Return to the beginning of the tape"
  More complicated but we have several options:
  1. Mark the first cell. E.g., for each $t \in \Sigma$, include both t and $\dot{t}$ in Γ; start the TM by replacing the first cell with its dotted version
  2. Copy the input right one cell and put a special marker like \$ in the first cell
  3. Use the fact that when TMs try to move left in the left-most cell, they stay in place: Remember the current symbol (by using states), replace it with \$, move left, if the cell remains \$, it's at the left-most cell. In any case, replace the original cell's symbol

Copying input right one cell

Consider $\Sigma = \{a, b\}$ and a TM that copies its input one cell right, replacing the first cell with \$ returns to the beginning and accepts

w = aaba



a	a	b	a		
↑					

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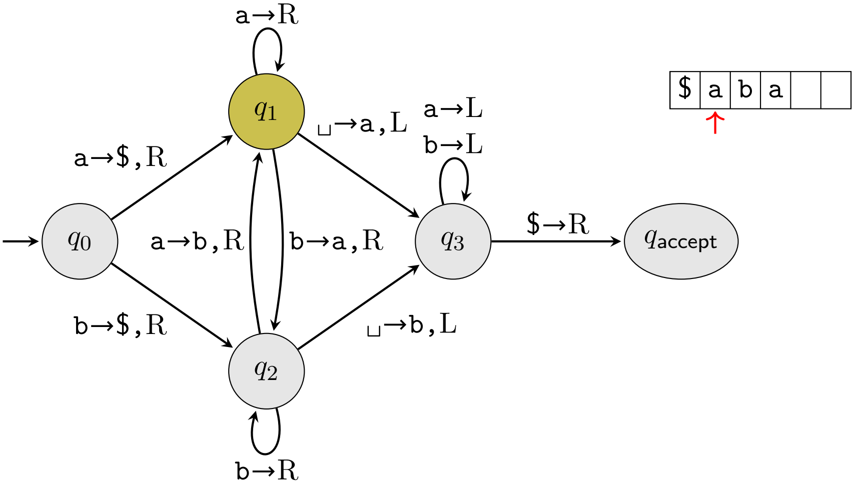

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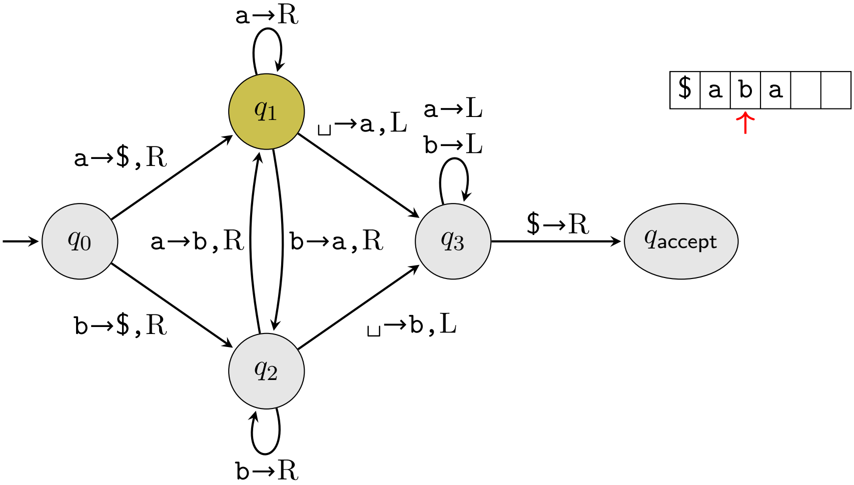

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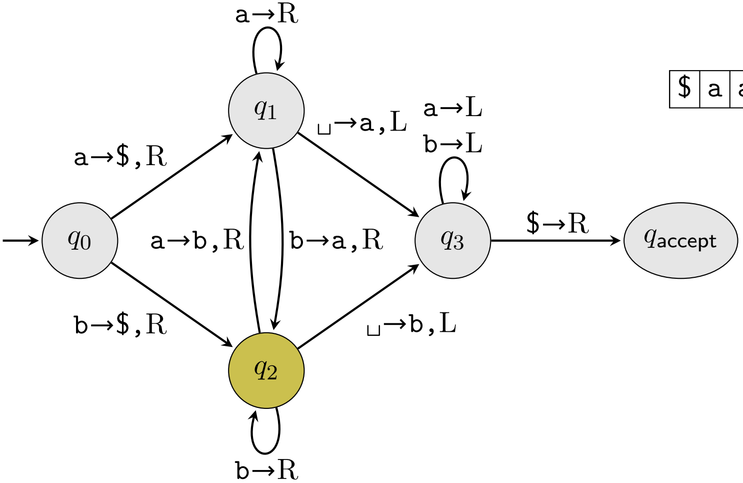

\$	a	a	a		
			↑		

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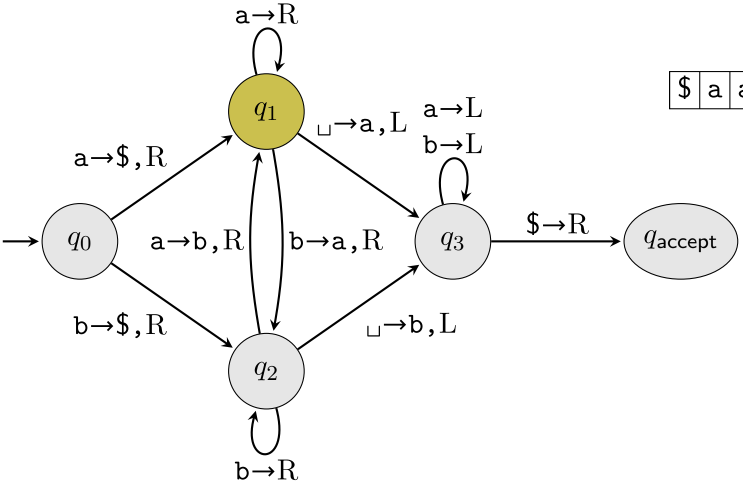

\$	a	a	b		

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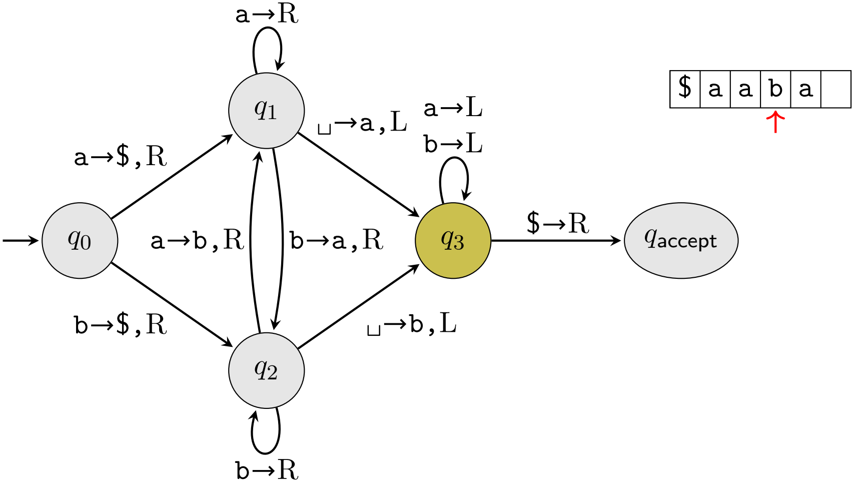

\$	a	a	b	a	

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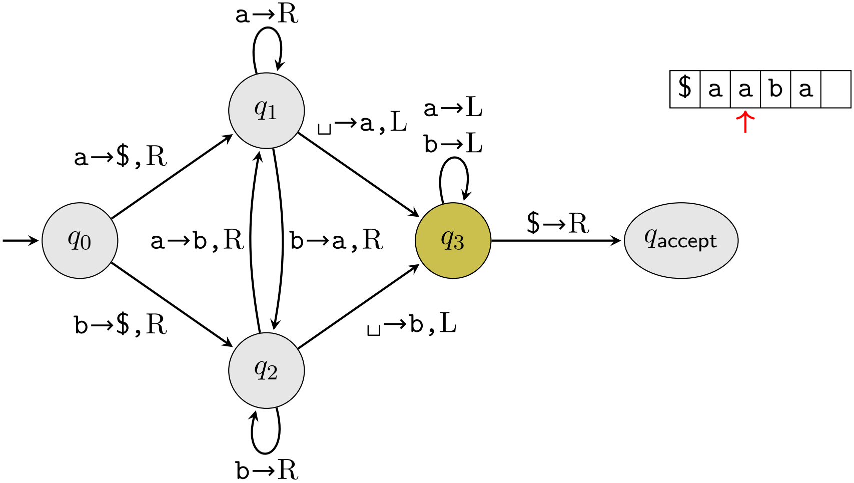

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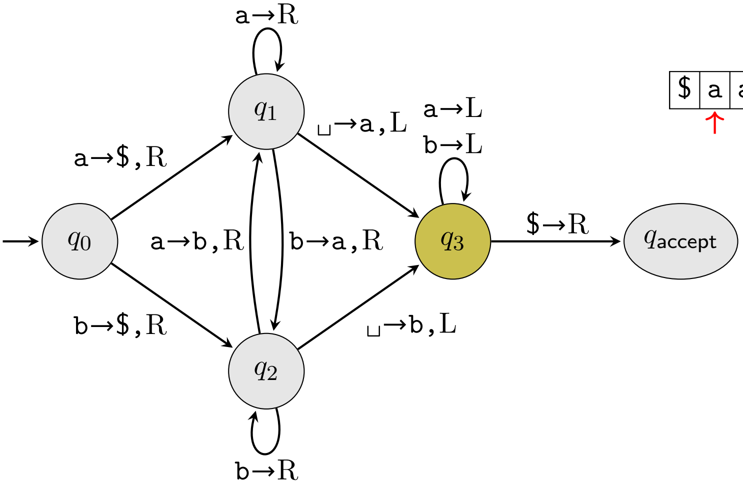

\$	a	a	b	a	
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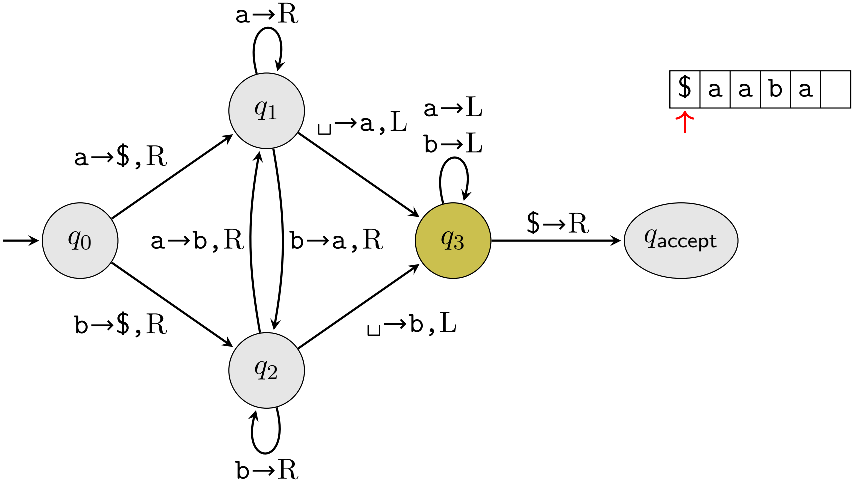

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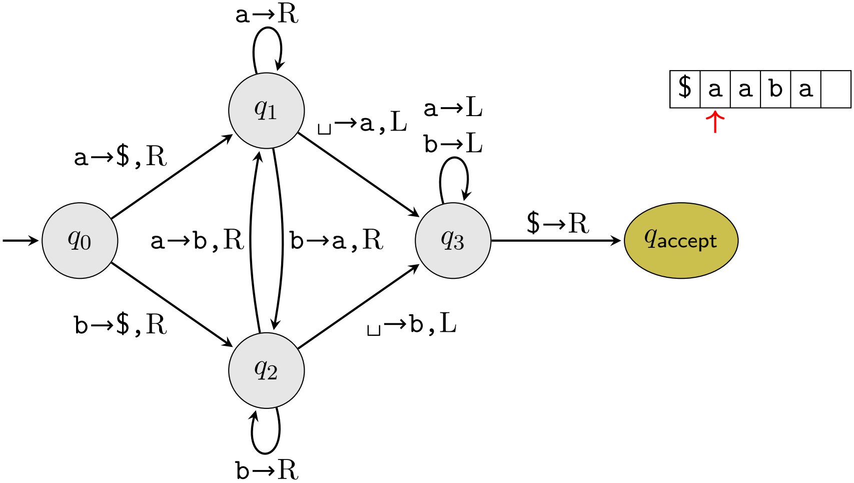

\$	a	a	b	a	

Returning to the beginning of the tape

Consider $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and a TM that scans right to the first blank, and then returns to the first cell and accepts

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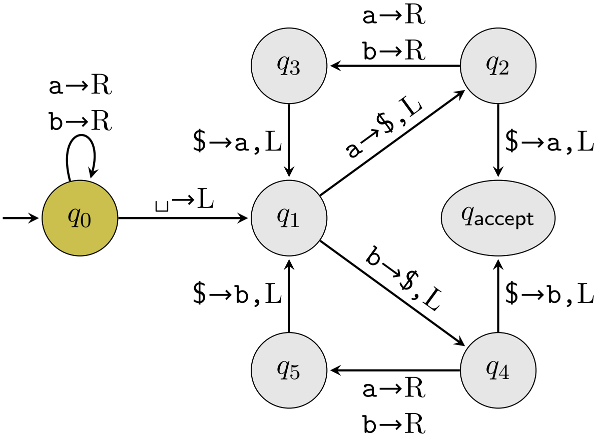

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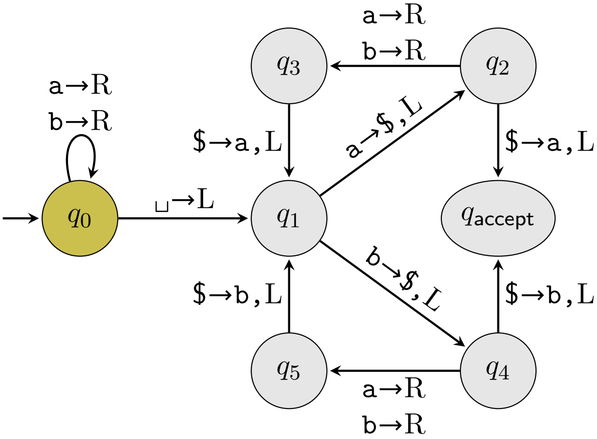

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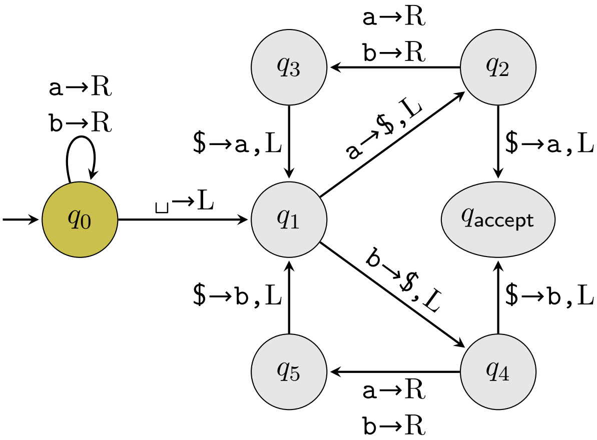

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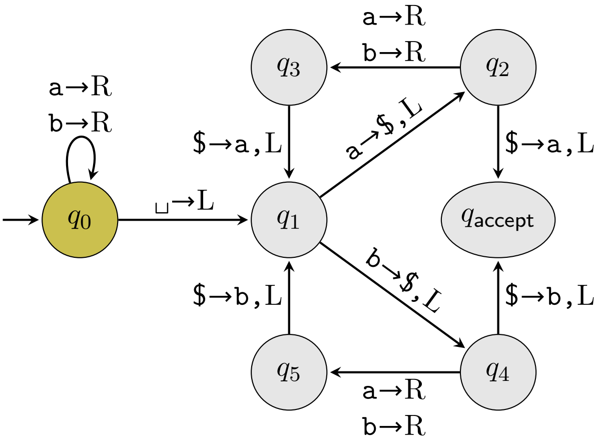

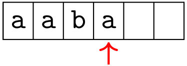

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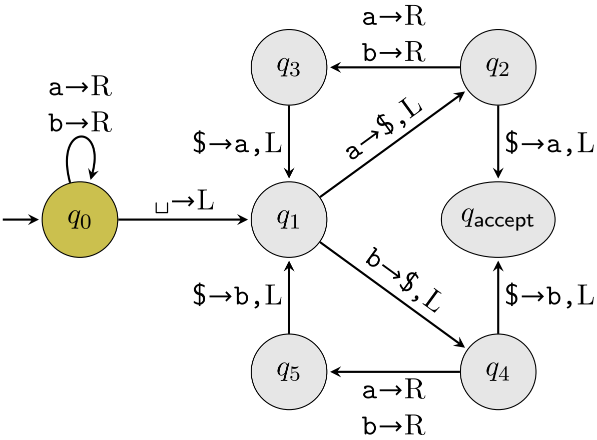

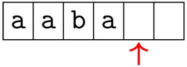

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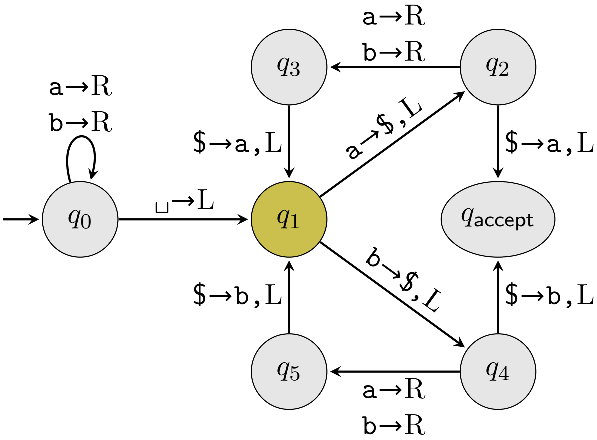

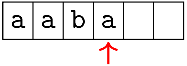

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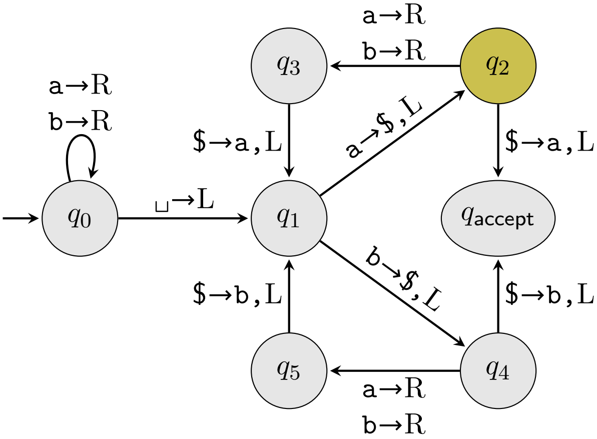

a	a	b	$		

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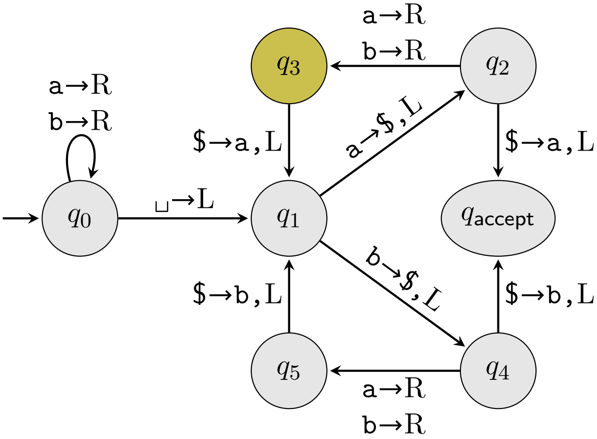

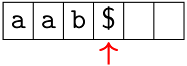

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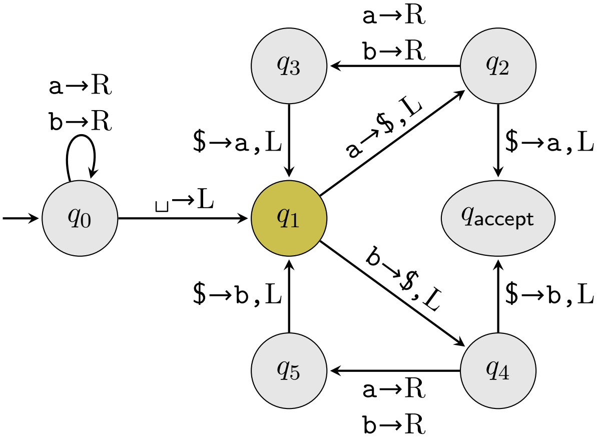

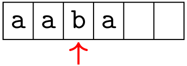

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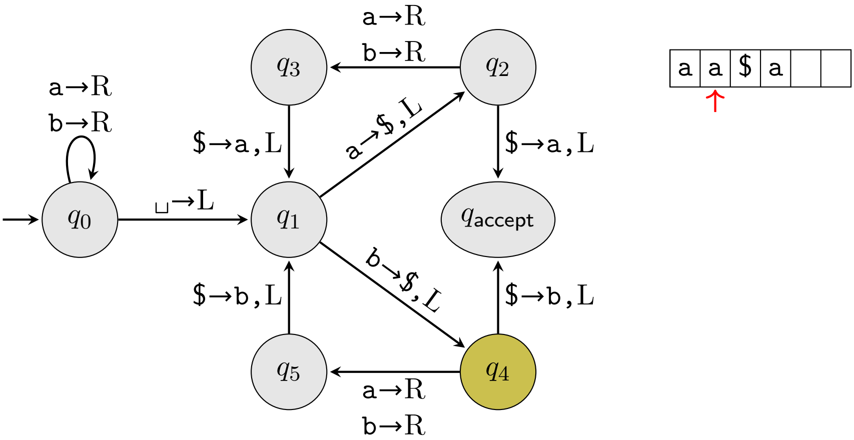

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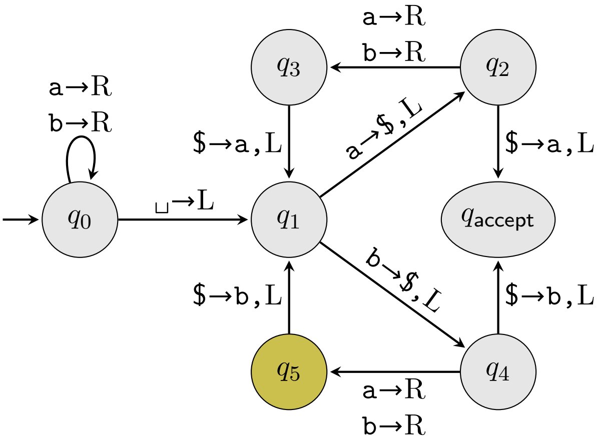

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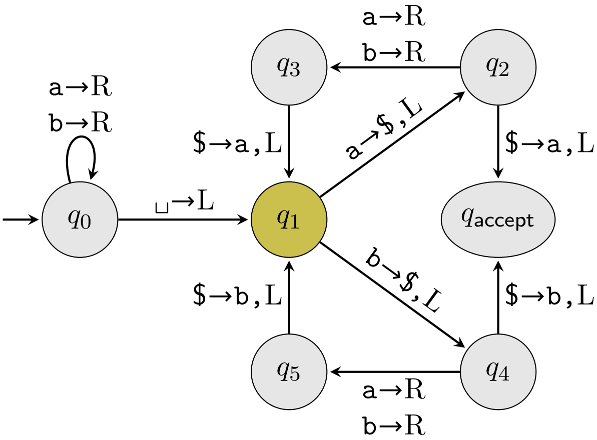

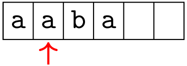

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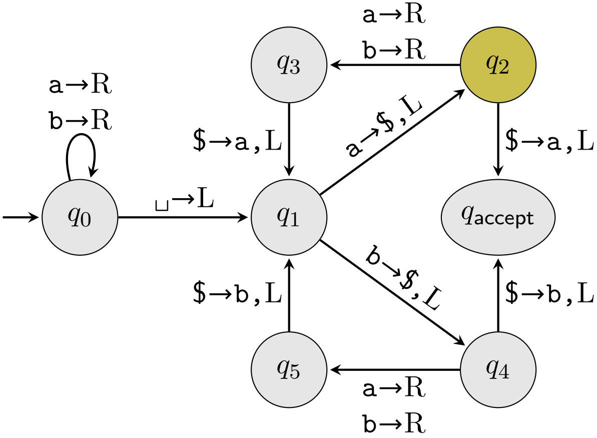

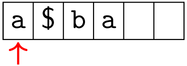

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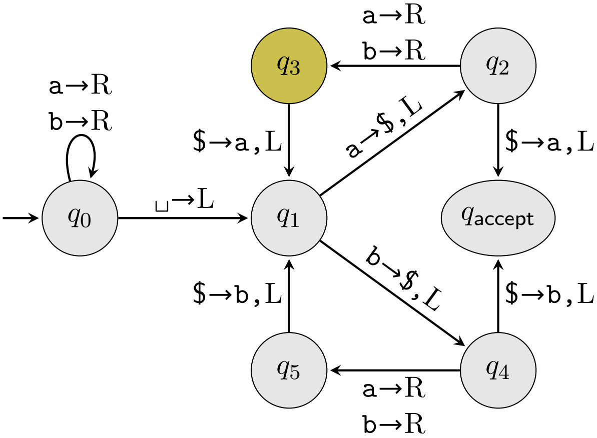

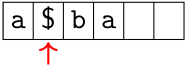

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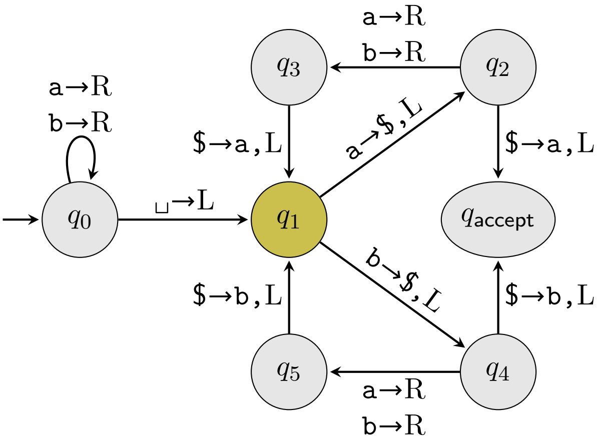

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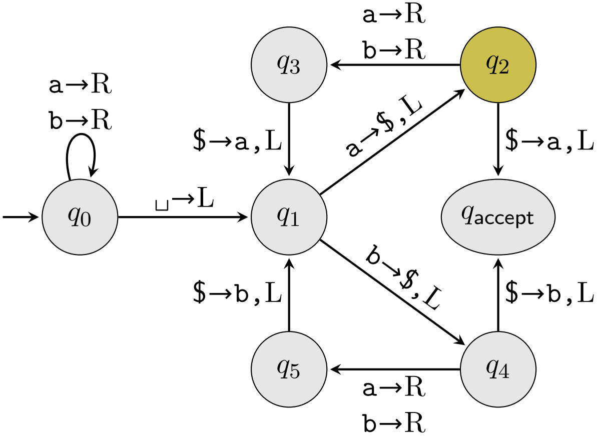

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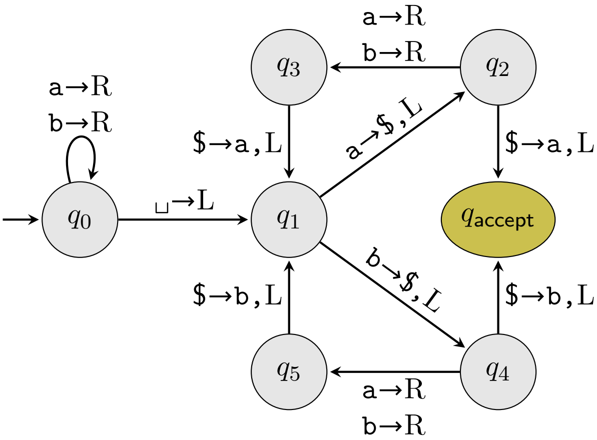

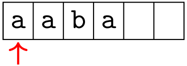

a	a	b	a		

Equal number of as and bs

$C = \{w \mid w \in \{\mathtt{a}, \mathtt{b}\}^*$ has the same number of as and bs$\}$

M = "On input w,

1. Scan right and mark the first unmarked a, if none exist, goto step 4
2. Return the head to the beginning of the tape, scan right and mark the first unmarked b. If none exist, reject
3. Return the head to the beginning of the tape and goto step 1
4. Return the head to the beginning of the tape and scan right. If any unmarked bs exist, reject, otherwise accept."

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4. Return the head to the beginning of the tape and scan right. If any unmarked bs exist, reject, otherwise accept."

We can easily implement each of these steps using a few states in a TM, but this is much clearer

What you *cannot* do

Each step in an implementation-level description (or a higher-level description) of a TM must be realizable in a real TM

You cannot have steps like

- "Check if w is in the language and accept, otherwise reject" (unless the language in question here is regular or other classes of languages we've shown are decidable)
- "If M loops, then ." There's no way, in general, to determine if a TM is looping
- Equivalently, "If M doesn't halt then ."

We'll see more examples of things TM's can't do later