CS 383

Lecture 13 – Closure properties of context-free languages

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Spring 2024

CFLs and PDAs

Theorem

Every context-free language can be recognized by some PDA.

Proof idea.

1. Use the PDA's stack to perform a left-most derivation of a word in the language
2. Match the PDA's input symbols against the stack, popping each one
3. Accept if stack is empty when there's no more input

What we'd like to do

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A left-most derivation of the string abaaa is

$$S \Rightarrow \mathtt{a}S\mathtt{a} \Rightarrow \mathtt{ab}T\mathtt{aa} \Rightarrow \mathtt{aba}T\mathtt{aa} \Rightarrow \mathtt{abaaa}.$$

We want to start by pushing S on the stack, then performing the derivation step by step so that abaaa ends on the stack, and then match the input

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We want to start by pushing S on the stack, then performing the derivation step by step so that abaaa ends on the stack, and then match the input

There are two complications

1. The first step in the derivation $S \Rightarrow \mathtt{a}S\mathtt{a}$ requires popping one symbol and pushing three
2. We can only replace symbols at the top of the stack

Pushing multiple symbols

We would like to write a transition like $q \xrightarrow{\varepsilon,\, S \to \mathtt{a}T\mathtt{b}} r$

but $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to P(Q \times \Gamma_\varepsilon)$ doesn't allow that

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Instead, use multiple transitions $q \xrightarrow{\varepsilon, S \to \mathtt{b}} \bigcirc \xrightarrow{\varepsilon, \varepsilon \to T} \bigcirc \xrightarrow{\varepsilon, \varepsilon \to \mathtt{a}} r$
Note that the symbols are pushed on in reverse order

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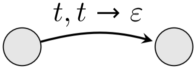
Rather than first deriving the whole string on the stack and then matching the input,

- If the top of the stack is a terminal, match it to the next input symbol

  for each $t \in \Sigma$
- If the top of the stack is a variable, replace it with the RHS of a corresponding rule

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  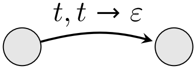

  for each $t \in \Sigma$

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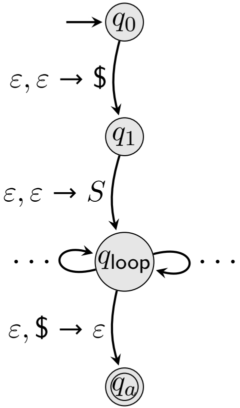
In fact, we only need four main states plus any additional states necessary to push multiple symbols

The q_{loop} state is where all the real work happens

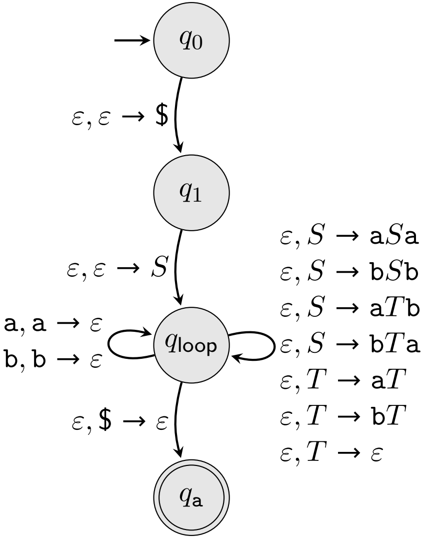


Example

$S \to \mathtt{a}S\mathtt{a} \mid \mathtt{b}S\mathtt{b} \mid \mathtt{a}T\mathtt{b} \mid \mathtt{b}T\mathtt{a}$

$T \to \mathtt{a}T \mid \mathtt{b}T \mid \varepsilon$

1. For each $t \in \Sigma$, add the transition $t, t \to \varepsilon$ from q_{loop} to q_{loop}
2. For each rule $A \to u_1 u_2 \cdots u_n$ for $u_i \in V \cup \Sigma$, add $n-1$ new states (if $n > 1$) and transitions to pop A and push $u_1, u_2, \ldots, u_n$ on in reverse order

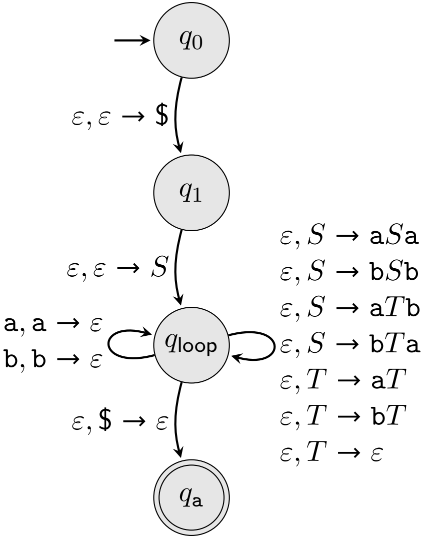


[The rules on the right need 10 extra states to make this a proper PDA]

Running the PDA on some input

Consider running the PDA on the input abaaa. The stack is shown on the right after each step

1. push \$; \$



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1. push \$; $\quad$ \$
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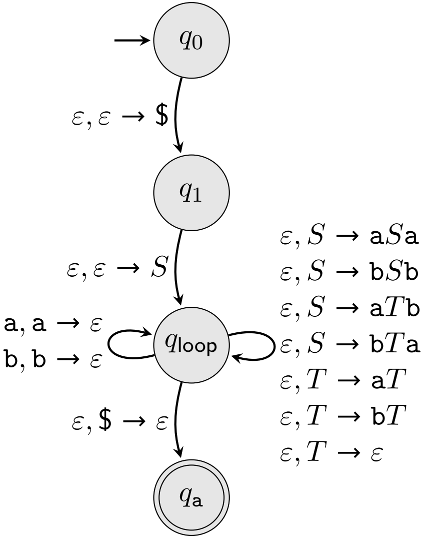

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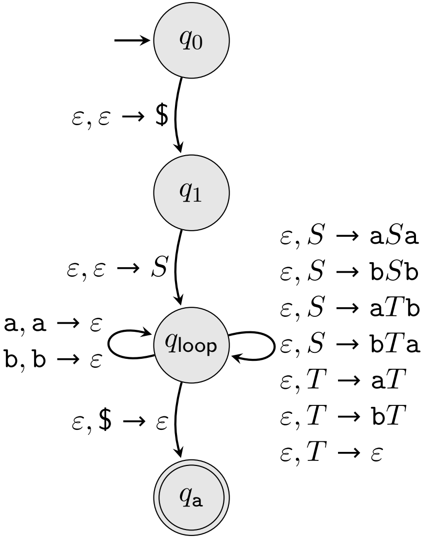
1. push \$; — \$
2. push S; — $S\$$
3. pop S, push aSa; — aSa\$



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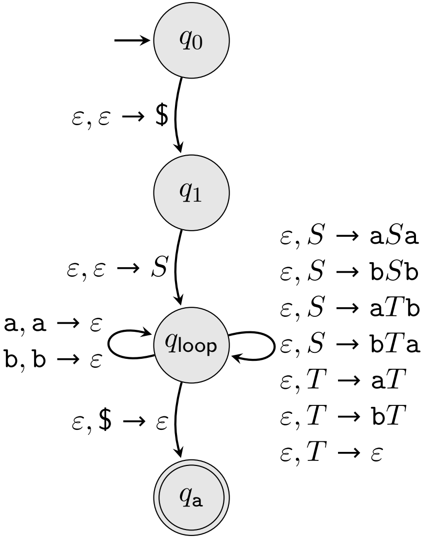

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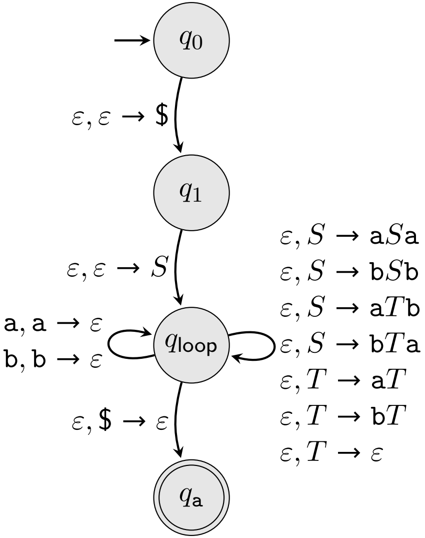

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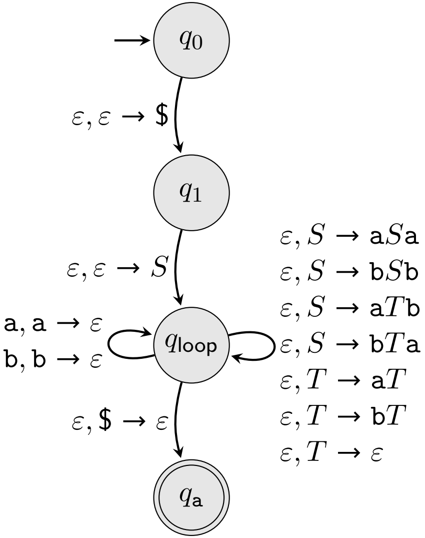

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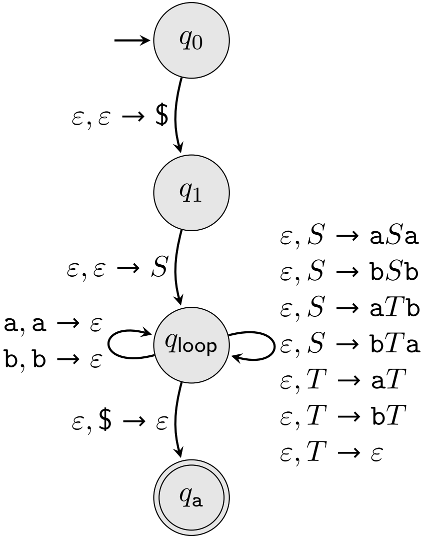

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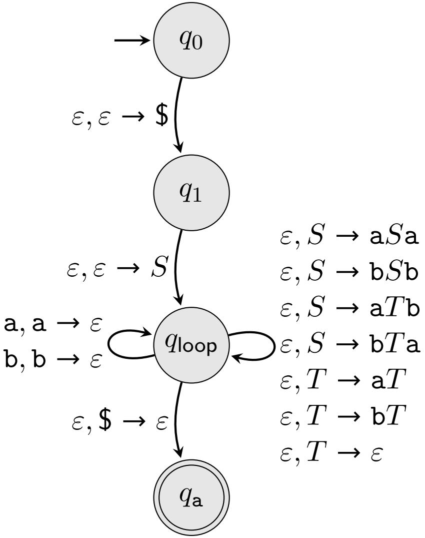

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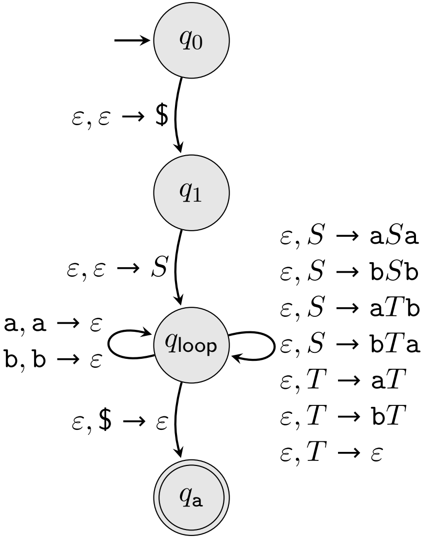

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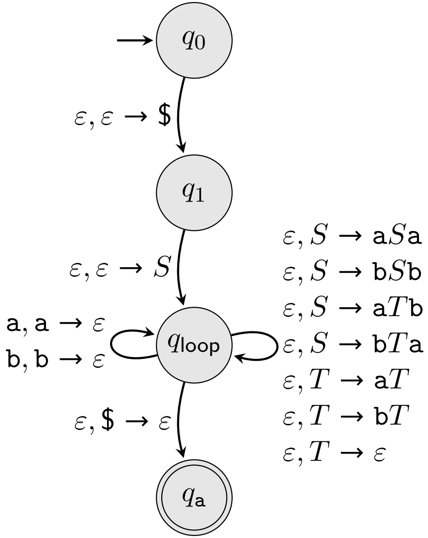

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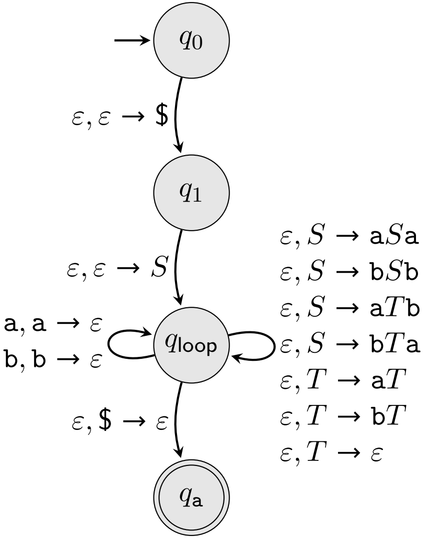

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9. pop T, push ε;	aa\$
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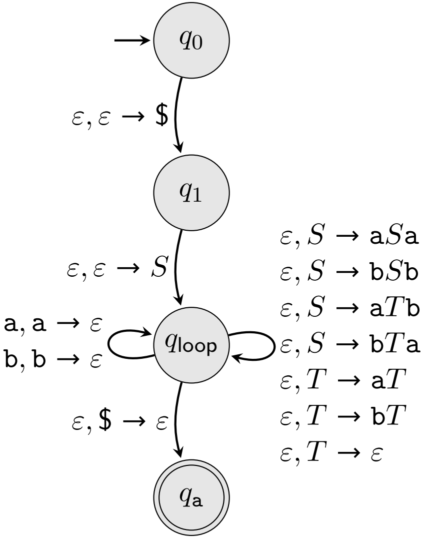
11. read and pop a;	\$
12. pop \$ and accept;	ε



Proving that every CFL is recognized by a PDA

Proof.

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Let A be a CFL generated by a CFG $G = (V, \Sigma, R, S)$.

Construct the PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, \{q_a\})$ with states $Q = \{q_0, q_1, q_{\text{loop}}, q_a\} \cup E$ where E are the extra states we need for each rule and $\Gamma = V \cup \Sigma \cup \{\$\}$.

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Start with the transitions

$\varepsilon, \varepsilon \to \$$ from q_0 to q_1,

$\varepsilon, \varepsilon \to S$ from q_1 to q_{loop}, and

$\varepsilon, \$ \to \varepsilon$ from q_{loop} to q_a

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For each $t \in \Sigma$, add the transition $t, t \to \varepsilon$ from q_{loop} to q_{loop}.

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$\varepsilon, \varepsilon \to S$ from q_1 to q_{loop}, and
$\varepsilon, \$ \to \varepsilon$ from q_{loop} to q_a

For each $t \in \Sigma$, add the transition $t, t \to \varepsilon$ from q_{loop} to q_{loop}.

For each rule $A \to u$ add the states and transitions necessary to pop A and push u in reverse order from q_{loop} to q_{loop}.

Proof continued

Consider running M on input $w = w_1 w_2 \cdots w_n$ for $w_i \in \Sigma$.

The first time M enters state q_{loop}, the stack is $S\$$ and no input has been read.

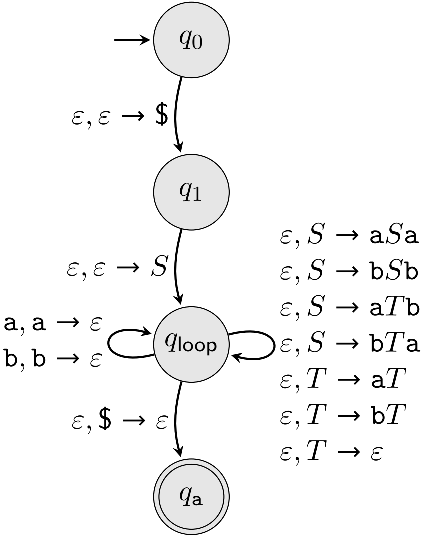
Every subsequent time it enters q_{loop}, the input read so far concatenated with the stack is a step in some left-most derivation of w (followed by a \$).

I.e., if k symbols have been read from the input and the stack is s, then $w_1 w_2 \cdots w_k s$ is a step in the derivation of w

Returning to the example

$$S \Rightarrow \mathtt{a}S\mathtt{a} \Rightarrow \mathtt{ab}T\mathtt{aa} \Rightarrow \mathtt{aba}T\mathtt{aa} \Rightarrow \mathtt{abaaa}$$

State	Action	Input read	Stack
q_0	push \$	ε	\$



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q_0	push \$	ε	\$
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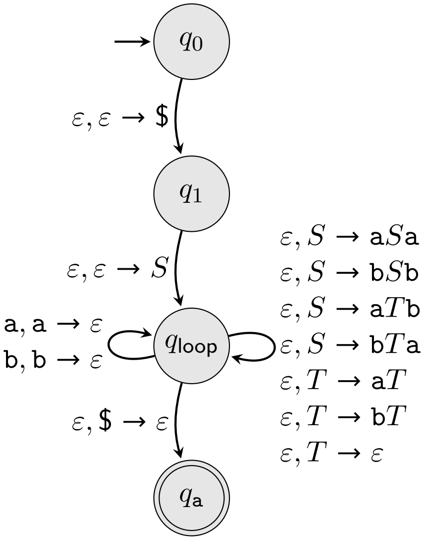

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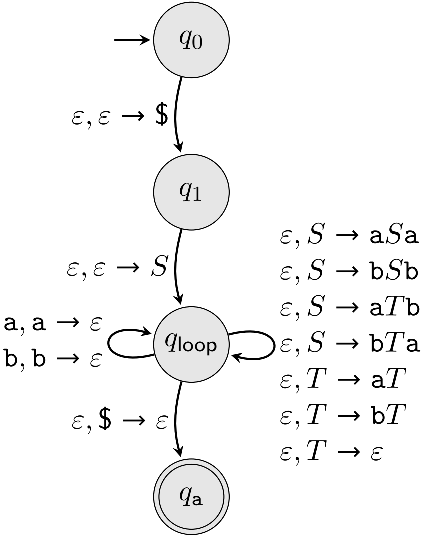
State	Action	Input read	Stack
q_0	push \$	ε	\$
q_1	push S	ε	$S\$$
q_{loop}	pop S, push $\mathtt{a}S\mathtt{a}$	ε	$\mathtt{a}S\mathtt{a}\$$



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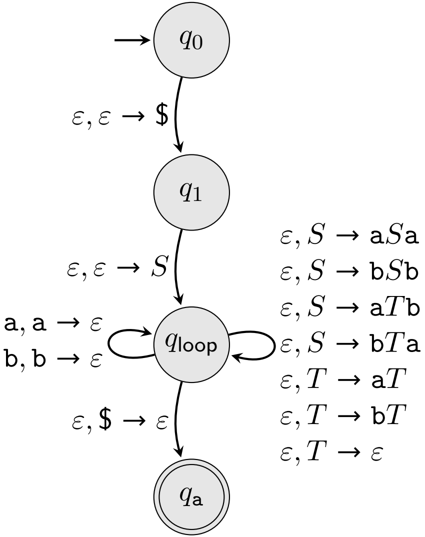
State	Action	Input read	Stack
q_0	push \$	ε	\$
q_1	push S	ε	S\$
q_{loop}	pop S, push aSa	ε	aSa\$
q_{loop}	read and pop a	a	Sa\$



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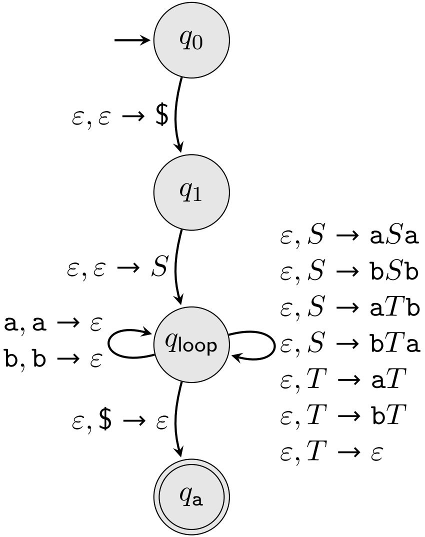
State	Action	Input read	Stack
q_0	push \$	ε	\$
q_1	push S	ε	S\$
q_{loop}	pop S, push $\mathtt{a}S\mathtt{a}$	ε	$\mathtt{a}S\mathtt{a}$\$
q_{loop}	read and pop a	a	$S\mathtt{a}$\$
q_{loop}	pop S, push $\mathtt{b}T\mathtt{a}$	a	$\mathtt{b}T\mathtt{aa}$\$



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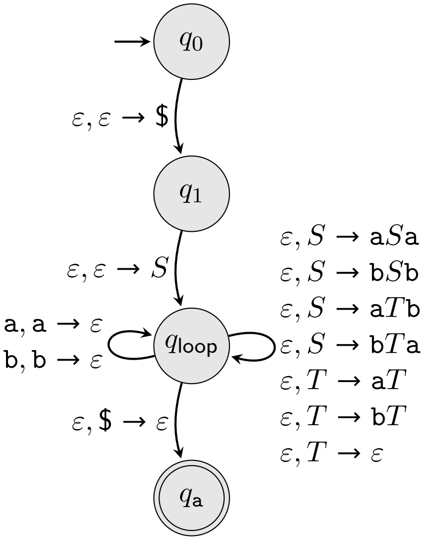
q_{loop}	pop S, push $\mathtt{a}S\mathtt{a}$	ε	$\mathtt{a}S\mathtt{a}\$$
q_{loop}	read and pop a	a	$S\mathtt{a}\$$
q_{loop}	pop S, push $\mathtt{b}T\mathtt{a}$	a	$\mathtt{b}T\mathtt{aa}\$$
q_{loop}	read and pop b	ab	$T\mathtt{aa}\$$



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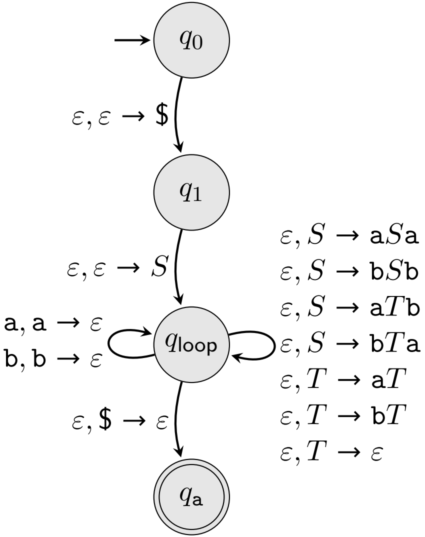

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q_{loop}	pop T, push $\mathtt{a}T$	ab	$\mathtt{a}T\mathtt{aa}$\$
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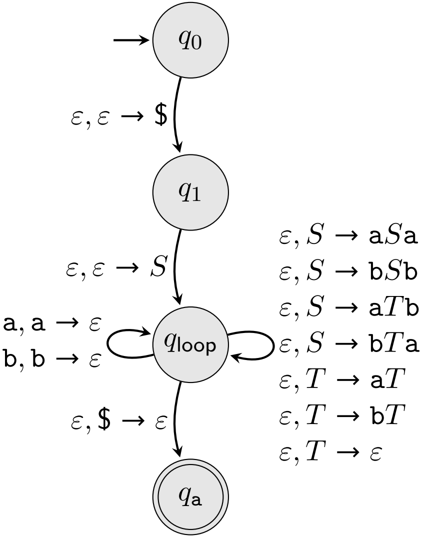

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q_{loop}	pop T, push aT	ab	aTaa\$
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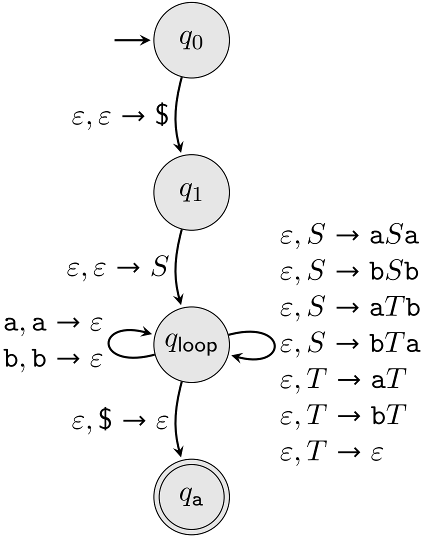

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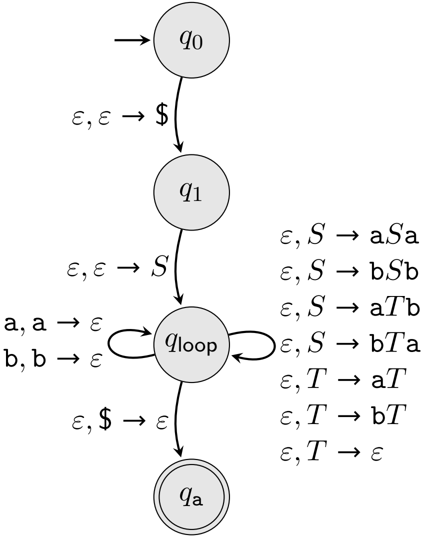
q_{loop}	read and pop b	ab	Taa\$
q_{loop}	pop T, push aT	ab	aTaa\$
q_{loop}	read and pop a	aba	Taa\$
q_{loop}	pop T, push ε	aba	aa\$
q_{loop}	read and pop a	abaa	a\$



Returning to the example

$$S \Rightarrow \mathtt{a}S\mathtt{a} \Rightarrow \mathtt{ab}T\mathtt{aa} \Rightarrow \mathtt{aba}T\mathtt{aa} \Rightarrow \mathtt{abaaa}$$

State	Action	Input read	Stack
q_0	push \$	ε	\$
q_1	push S	ε	S\$
q_{loop}	pop S, push aSa	ε	aSa\$
q_{loop}	read and pop a	a	Sa\$
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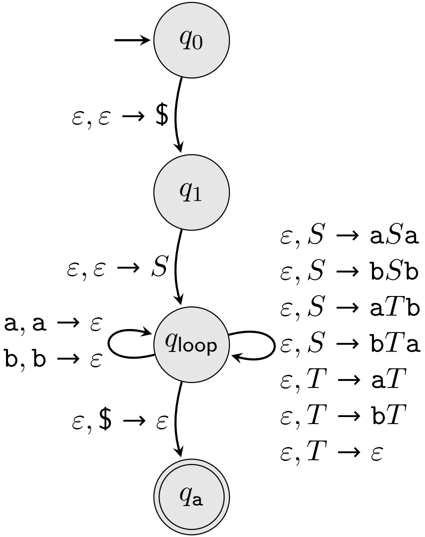

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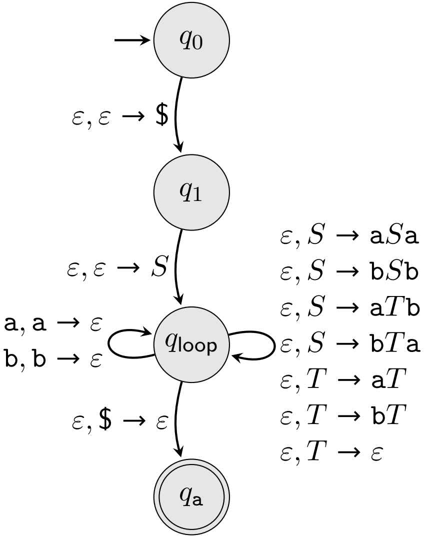
q_{loop}	pop \$	abaaa	ε



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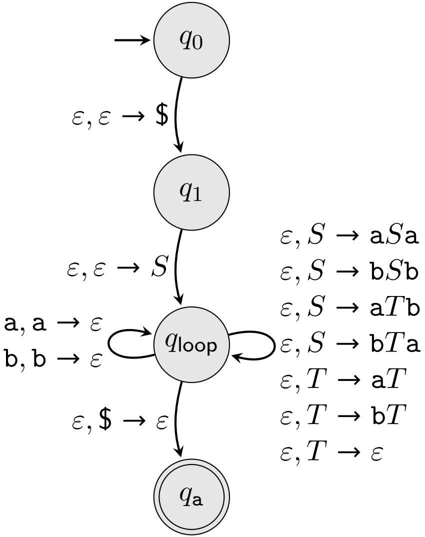
q_{loop}	read and pop a	abaaa	\$
q_{loop}	pop \$	abaaa	ε
q_{a}	accept	abaaa	ε



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Consider running M on input $w = w_1 w_2 \cdots w_n$ for $w_i \in \Sigma$.

The first time M enters state q_{loop}, the stack is S\$ and no input has been read.

Every subsequent time it enters q_{loop}, the input read so far concatenated with the stack is a step in some left-most derivation of w (followed by a \$).

I.e., if k symbols have been read from the input and the stack is s, then $w_1 w_2 \cdots w_k s$ is a step in the derivation of w

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M accepts w once the derivation is complete and all terminals have been matched. Therefore, each string accepted by M is in A.

For each $w \in A$, there is some left-most derivation of w by G. By construction, M performs the derivation on the stack while matching leading terminals.

Thus $L(M) = A$. $\square$

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Theorem

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Proof idea.

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   - has rules $\langle p, q \rangle \to a \langle r, s \rangle b$ for $p, q, r, s \in Q$ and $a, b \in \Sigma_\varepsilon$ if $(r, u) \in \delta(p, a, \varepsilon)$ and $(q, \varepsilon) \in \delta(s, b, u)$

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3. Prove (by induction) that each variable $\langle q, r \rangle$ has the property $\langle q, r \rangle \overset{*}{\Rightarrow} x \in \Sigma^*$ iff starting M in state q with an empty stack and running on input x causes M to move to state r and end with an empty stack

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4. Conclude that $\langle q_0, q_a\rangle \overset{*}{\Rightarrow} w$ iff $w \in L(M)$

Closure properties of CFLs

The class of context-free languages is closed under

- Union
- Concatenation
- Kleene star
- PREFIX
- SUFFIX
- Reversal
- Intersection with a regular language
- Quotient by a string
- Quotient by a regular language

We proved closure under union, concatenation, Kleene star, and PREFIX previously

Reversal

Theorem

Context-free languages are closed under reversal.

Proof. Let B be a context-free language generated by a CFG $G = (V, \Sigma, R, S)$.

Construct CFG $G' = (V, \Sigma, R', S)$ where

$$R' = \{A \to u^{\mathcal{R}} \mid A \to u \text{ is a rule in } R\}.$$

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To prove that $L(G') = B^{\mathcal{R}}$, we want to show that for each variable $A \in V$ and $u \in (V \cup \Sigma)^*$, $A \overset{*}{\Rightarrow}_G u$ in n steps iff $A \overset{*}{\Rightarrow}_{G'} u^{\mathcal{R}}$ in n steps.

Let's write $\overset{k}{\Rightarrow}$ to mean $\overset{*}{\Rightarrow}$ in exactly k steps.

Proof continued

Base case $n = 0$. If $A \overset{0}{\Rightarrow}_G u$, then $u = u^{\mathcal{R}} = A$ so $A \overset{0}{\Rightarrow}_{G'} u^{\mathcal{R}}$, and vice versa.

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If $A \overset{n+1}{\Rightarrow}_G u$, then there is some $C \in V$ and $x, y, z \in (V \cup \Sigma)^*$ such that $u = xyz$, $A \overset{n}{\Rightarrow}_G xCz$, and $C \Rightarrow_G y$.

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By the inductive hypothesis $A \overset{n}{\Rightarrow}_{G'} z^{\mathcal{R}}Cx^{\mathcal{R}}$ and by construction $C \Rightarrow_{G'} y^{\mathcal{R}}$. Thus $A \overset{n+1}{\Rightarrow}_{G'} z^{\mathcal{R}}y^{\mathcal{R}}x^{\mathcal{R}} = (xyz)^{\mathcal{R}} = u^{\mathcal{R}}$. Swapping G and G' shows the converse.

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Thus, $A \overset{n+1}{\Rightarrow}_G u$ iff $A \overset{n+1}{\Rightarrow}_{G'} u^{\mathcal{R}}$.

Therefore, for $w \in B$, $S \overset{*}{\Rightarrow}_G w$ iff $S \overset{*}{\Rightarrow}_{G'} w^{\mathcal{R}}$ so $L(G') = B^{\mathcal{R}}$. $\square$

Suffix

Theorem

Context free languages are closed under $\textsc{Suffix}$.

Proof.

Since $\textsc{Suffix}(A) = \textsc{Prefix}((A^{\mathcal{R}})^{\mathcal{R}})$ and CFLs are closed under reversal and $\textsc{Prefix}$, CFLs are closed under $\textsc{Suffix}$. $\square$

Intersection of a CFL and a regular language

Theorem

The intersection of a CFL and a regular language is context-free.

Proof.

Let A be a CFL recognized by the PDA $M_1 = (Q_1, \Sigma, \Gamma, \delta_1, q_1, F_1)$ and B be a regular language recognized by the NFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$.

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Construct the PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ where

$$\begin{aligned} Q &= Q_1 \times Q_2 \\ q_0 &= (q_1, q_2) \\ F &= F_1 \times F_2 \\ \delta\big((q,r),a,b\big) &= \big\{\big((s,t),c\big) \bigm| (s,c) \in \delta_1(q,a,b) \text{ and } t \in \delta_2(r,a)\big\} \quad \text{for } a \in \Sigma_\varepsilon,\ b,c \in \Gamma_\varepsilon \end{aligned}$$

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As M runs on input w, its stack and the first element of its state change according to δ_1 whereas the second element of its state changes according to δ_2.

M accepts w iff M_1 accepts w and M_2 accepts w. Therefore, $L(M) = A \cap B$. □

What about intersection with another CFL?

Are context-free languages closed under intersection?

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Consider $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and

$$A = \{\mathtt{a}^m \mathtt{b}^m \mathtt{c}^n \mid m, n \geq 0\}$$
$$B = \{\mathtt{a}^m \mathtt{b}^n \mathtt{c}^n \mid m, n \geq 0\}$$

Both B and C are context-free. Is

$$A \cap B = \{\mathtt{a}^n \mathtt{b}^n \mathtt{c}^n \mid n \geq 0\}?$$

How can we keep track of how many as and bs we've seen to ensure we get the same number of cs using a PDA?

How about trying to generate such strings with a CFG?

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Next time, we'll see that $B \cap C$ is *not* context-free!