CS 383

Lecture 22 – Mapping reductions

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Review of decidable languages

- Context-free languages (and thus regular)

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- Acceptance problems
  - A_{DFA}
  - A_{NFA}
  - A_{REX}
  - A_{CFG}

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- Equivalence problems
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- EQ_{TM}
- $\textsc{Regular}_{\mathsf{TM}}$

Turing recognizable (RE) and co-Turing-recognizable (coRE)

Recall, L is decidable iff L is RE and coRE

Language	RE	coRE
A_{DFA}	✔	✔
E_{DFA}	✔	✔
EQ_{DFA}	✔	✔
A_{CFG}	✔	✔
E_{CFG}	✔	✔
EQ_{CFG}	✖	✔
$\textsc{Diag}$	?	?
A_{TM}	✔	✖
$\textsc{Halt}_{\text{TM}}$	?	?
E_{TM}	✖	✔
EQ_{TM}	?	?
$\textsc{Regular}_{\text{TM}}$	?	?

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Recall that A reduces to B (written $A \leq B$) means
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"If B is decidable, then A is decidable"

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Similarly, we proved some languages aren't coRE by showing that they're RE but not decidable

Reductions alone were not sufficient; we need a *stronger* notion of reduction

Computable functions

A function $f : \Sigma^* \to \Sigma^*$ is a computable function if there is some TM M such that when M is run on w, M halts with $f(w)$ on the tape (and nothing else)

This is similar to a decider in that M cannot loop, but there's no notion of accepting or rejecting a string, M just computes a function

Examples of computable functions

- Arithmetic: $\langle k, m, n \rangle \mapsto \langle k \cdot m - 67n \rangle$ where $k, m, n \in \mathbb{Z}$
  The corresponding TM performs the arithmetic and then copies the result to the beginning of the tape and clears the rest

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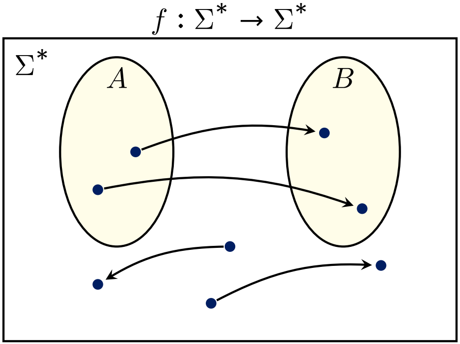
Anything that a TM can do without looping, including running deciders, is permissible

If the form of the input is wrong (e.g., if the TM is expecting $\langle M, w \rangle$ but gets something else), then it clears the tape and halts (i.e., outputs ε)

Mapping reducibility

Language A is mapping reducible to language B, written $A \leq_m B$, if there exists a computable function $f : \Sigma^* \to \Sigma^*$ such that for each $w \in \Sigma^*$,

$$w \in A \iff f(w) \in B$$



f maps elements of A to elements of B
f maps elements of $\overline{A}$ to elements of $\overline{B}$

Mapping instances of problems to instances of other problems

Consider the problems

1. Is the string w recognized by the PDA P?
2. Is the string x generated by the CFG G?

We express both of these as languages, A_{PDA} and A_{CFG}, respectively

An instance of the first problem is the (representation of the) pair $\langle P, w\rangle$ and an instance of the second problem is $\langle G, x\rangle$

A mapping reduction $A \leq_{\mathrm{m}} B$ takes an instance of problem A and maps it to an instance of problem B such that the solution to the latter gives the solution to the former

E.g., $\langle P, w\rangle \mapsto \langle G, w\rangle$ where $L(G) = L(P)$ is a computable mapping and $\langle P, w\rangle \in A_{\mathsf{PDA}} \iff \langle G, w\rangle \in A_{\mathsf{CFG}}$ so $A_{\mathsf{PDA}} \leq_{\mathrm{m}} A_{\mathsf{CFG}}$

Question 1

Is $A_{\mathsf{CFG}} \leq_{\mathrm{m}} A_{\mathsf{PDA}}$?

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Yes. The mapping $\langle G, w \rangle \mapsto \langle P, w \rangle$ where $L(P) = L(G)$ is computable because the CFG to PDA conversion is a simple algorithm.

As before, $\langle G, w \rangle \in A_{\mathsf{CFG}} \iff \langle P, w \rangle \in A_{\mathsf{PDA}}$

Question 2

Is $A_{\mathsf{DFA}} \leq_{\mathrm{m}} A_{\mathsf{CFG}}$?

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Yes. We can convert a DFA to an equivalent CFG; i.e., $\langle M, w\rangle \mapsto \langle G, w\rangle$ where $L(G) = L(M)$ is computable and clearly $\langle M, w\rangle \in A_{\mathsf{DFA}} \iff \langle G, w\rangle \in A_{\mathsf{CFG}}$

Question 3

Is $A_{\mathsf{CFG}} \leq_{\mathrm{m}} A_{\mathsf{DFA}}$?

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Is $A_{\text{CFG}} \leq_{\mathrm{m}} A_{\text{DFA}}$?

Perhaps counterintuitively, yes!

Remember, A_{CFG} is decidable so we can use the decider R for it when constructing our mapping
T = "On input $\langle G, w \rangle$,

1. Run R on $\langle G, w \rangle$
2. If R accepts, let M be the 1-state DFA such that $L(M) = \Sigma^*$
3. If R rejects, let M be the 1-state DFA such that $L(M) = \varnothing$
4. Output $\langle M, \varepsilon \rangle$"

This won't loop because R is a decider.

If $\langle G, w \rangle \in A_{\text{CFG}}$, then $L(M) = \Sigma^*$ so $\langle M, \varepsilon \rangle \in A_{\text{DFA}}$

If $\langle G, w \rangle \notin A_{\text{CFG}}$, then $L(M) = \varnothing$ so $\langle M, \varepsilon \rangle \notin A_{\text{DFA}}$

Mapping reductions are a stronger form of reduction

What we've called a reduction up until now is also called a Turing reduction

Theorem

If $A \leq_{\mathrm{m}} B$*, then* $A \leq B$*. In other words, if* $A \leq_{\mathrm{m}} B$ *and* B *is decidable, then* A *is decidable*

How can we prove this?

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Theorem

If $A \le_{\mathrm{m}} B$, then $A \le B$. In other words, if $A \le_{\mathrm{m}} B$ and B is decidable, then A is decidable

How can we prove this?

Proof.

Let R be a decider for B and let $f : \Sigma^* \to \Sigma^*$ be the mapping reduction.

D = "On input w,

1. Compute $f(w)$
2. Run R on $f(w)$ and if R accepts, then *accept*; otherwise *reject*"

f is computable and R is a decider so D is a decider.

If $w \in A$, then $f(w) \in B$ so R and thus D will accept

If $w \notin A$, then $f(w) \notin B$ so R and thus D will reject □

Using mapping reductions to show languages are undecidable

Just like with Turing reductions, we have a simple corollary:

Theorem

If $A \leq_{\mathrm{m}} B$ and A is undecidable, then B is undecidable

We typically use this fact by giving a TM that computes the mapping reduction

T = "On input $\langle$an instance of problem $A\rangle$,

1. Construct an instance of problem B
2. Output $\langle$the instance of problem $B\rangle$"

Rather than accept or reject, the TM T corresponding to the mapping outputs the result

Example: $E_{\mathsf{TM}} \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$

Show that $E_{\mathsf{TM}} \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$ by giving a TM T that computes the mapping
How do we do this?

Example: $E_{\mathsf{TM}} \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$

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Note that $\langle M \rangle$ is an instance of E_{TM} and $\langle M, M' \rangle$ is an instance of EQ_{TM}

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T = "On input $\langle M \rangle$,

1. Build TM M' such that $L(M') = \emptyset$
2. Output $\langle M, M' \rangle$"

Note that $\langle M \rangle$ is an instance of E_{TM} and $\langle M, M' \rangle$ is an instance of EQ_{TM}

We need to show that T doesn't loop and that $\langle M \rangle \in E_{\mathsf{TM}}$ iff $\langle M, M' \rangle \in EQ_{\mathsf{TM}}$

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T = "On input $\langle M \rangle$,

1. Build TM M' such that $L(M') = \varnothing$
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Note that $\langle M \rangle$ is an instance of E_{TM} and $\langle M, M' \rangle$ is an instance of EQ_{TM}

We need to show that T doesn't loop and that $\langle M \rangle \in E_{\mathsf{TM}}$ iff $\langle M, M' \rangle \in EQ_{\mathsf{TM}}$

Neither steps 1 nor 2 loop, so T doesn't loop

Next, we have a chain of iff

$$\langle M \rangle \in E_{\mathsf{TM}} \iff L(M) = \varnothing \iff L(M) = L(M') \iff \langle M, M' \rangle \in EQ_{\mathsf{TM}}$$

Example: $A_{\mathsf{TM}} \leq_{\mathrm{m}} \textsc{Halt}_{\mathsf{TM}}$

This one is more tricky: Given $\langle M, w \rangle$ (an instance of A_{TM}), we need to construct $\langle M', w \rangle$ such that M accepts w iff M' halts on w

How can we do this?

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T = "On input $\langle M, w\rangle$,

1. Construct a new TM M' = 'On input x,
   1. Run M on x
   2. If M accepts, then *accept*
   3. If M rejects, then *loop*'
2. Output $\langle M', w\rangle$"

Example: $A_{\mathsf{TM}} \leq_{\mathrm{m}} \textsc{Halt}_{\mathsf{TM}}$

This one is more tricky: Given $\langle M, w\rangle$ (an instance of A_{TM}), we need to construct $\langle M', w\rangle$ such that M accepts w iff M' halts on w

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2. Output $\langle M', w\rangle$"

Constructing the TM M' can't loop so T can't loop

If $\langle M, w\rangle \in A_{\mathsf{TM}}$, then M accepts w so M' accepts and thus halts on w so $\langle M', w\rangle \in \textsc{Halt}_{\mathsf{TM}}$

If $\langle M, w\rangle \notin A_{\mathsf{TM}}$, then either M rejects or loops on w and in either case, M' loops on w [why?] so $\langle M', w\rangle \notin \textsc{Halt}_{\mathsf{TM}}$

Example: $EQ_{\mathsf{CFG}} \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$

How do we show this?

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T = "On input $\langle G_1, G_2 \rangle$,

1. Construct TM M_1 s.t. $L(M_1) = L(G_1)$ (we can use the decider for A_{CFG} to do this)
2. Construct TM M_2 s.t. $L(M_2) = L(G_2)$
3. Output $\langle M_1, M_2 \rangle$"

Now what?

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2. Construct TM M_2 s.t. $L(M_2) = L(G_2)$
3. Output $\langle M_1, M_2 \rangle$"

Now what?

T can't loop because it's just constructing two TMs

Since $L(G_i) = L(M_i)$, $\langle G_1, G_2 \rangle \in EQ_{\mathsf{CFG}} \iff L(G_1) = L(G_2) \iff L(M_1) = L(M_2) \iff \langle M_1, M_2 \rangle \in EQ_{\mathsf{TM}}$

Mapping reductions between RE languages

Theorem
If $A \leq_{\mathrm{m}} B$ *and* B *is Turing-recognizable, then* A *is Turing-recognizable.*

How do we prove this?

Mapping reductions between RE languages

Theorem

If $A \leq_{\mathrm{m}} B$ and B is Turing-recognizable, then A is Turing-recognizable.

How do we prove this? Same construction as for the decidable case.

Proof.

Let R be a TM such that $L(R) = B$ and $f : \Sigma^* \to \Sigma^*$ be the computable mapping.
Build TM M to recognize A:
M = "On input w,

1. Run R on $f(w)$. If R accepts, then *accept*; if R rejects, then *reject*"

Now we just need to show that $L(M) = A$

$$w \in A \iff f(w) \in B \iff R \text{ accepts } f(w) \iff M \text{ accepts } w. \qquad \square$$

Proving that a language is not RE

Theorem

If $A \leq_{\mathrm{m}} B$ and A is not Turing-recognizable, then B is not Turing-recognizable

Why?

Proving that a language is not RE

Theorem

If $A \leq_{\mathrm{m}} B$ *and* A *is not Turing-recognizable, then* B *is not Turing-recognizable*

Why?

Proof.

If B were RE, then by the previous theorem, A would be RE. □

Mapping reduction between complements

Theorem

If $A \leq_{\mathrm{m}} B$, then $\overline{A} \leq_{\mathrm{m}} \overline{B}$ with the reduction given by the same mapping.

We just use the fact that if f is the computable mapping, then $w \in A \iff f(w) \in B$

Mapping reduction between complements

Theorem

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We just use the fact that if f is the computable mapping, then $w \in A \iff f(w) \in B$

Proof.

Let f be the mapping reduction from A to B. Then

$$w \in \overline{A} \iff w \notin A \iff f(w) \notin B \iff f(w) \in \overline{B}. \qquad \square$$

coRE

Theorem

If $A \leq_{\mathrm{m}} B$ and B is co-Turing-recognizable, then A is co-Turing-recognizable.

Why?

coRE

Theorem

If $A \leq_{\mathrm{m}} B$ and B is co-Turing-recognizable, then A is co-Turing-recognizable.

Why?

Proof.

By the previous theorem, $\overline{A} \leq_{\mathrm{m}} \overline{B}$.

Since B is coRE, $\overline{B}$ is RE and thus $\overline{A}$ is RE. Therefore, A is coRE. □

Not coRE

Theorem

If $A \leq_{\mathrm{m}} B$ and A is not co-Turing-recognizable, then B is not co-Turing-recognizable.

Proof.

If B were coRE, then A would be coRE by the previous theorem. □

Recapitulate our results

A and B are languages and $A \leq_{\mathrm{m}} B$.

Good-news reductions

- If B is decidable, then A is decidable
- If B is RE, then A is RE
- If B is coRE, then A is coRE

Bad-news reductions

- If A is not decidable, then B is not decidable
- If A is not RE, then B is not RE
- If A is not coRE, then B is not coRE

Example

Show $A_{\mathsf{TM}} \leq_{\mathrm{m}} \overline{E_{\mathsf{TM}}}$

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T = "On input $\langle M, w\rangle$,

1. Construct TM M_w = 'On input x,
    1. Ignore x and run M on w. If M accepts, then *accept*; if M rejects, then *reject*'
2. Output $\langle M_w\rangle$"

This is clearly computable (i.e., T doesn't loop)

Now we just need to show that $\langle M, w\rangle \in A_{\mathsf{TM}}$ iff $\langle M_w\rangle \in \overline{E_{\mathsf{TM}}}$

Example

Show $A_{\mathsf{TM}} \leq_{\mathrm{m}} \overline{E_{\mathsf{TM}}}$

We need to give a TM that takes as input an instance of A_{TM} and outputs an instance of $\overline{E_{\mathsf{TM}}}$

T = "On input $\langle M, w \rangle$,

1. Construct TM M_w = 'On input x,
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This is clearly computable (i.e., T doesn't loop)

Now we just need to show that $\langle M, w \rangle \in A_{\mathsf{TM}}$ iff $\langle M_w \rangle \in \overline{E_{\mathsf{TM}}}$

If $\langle M, w \rangle \in A_{\mathsf{TM}}$, then M accepts w so $L(M_w) = \Sigma^*$ and thus $\langle M_w \rangle \in \overline{E_{\mathsf{TM}}}$

If $\langle M, w \rangle \notin A_{\mathsf{TM}}$, then M doesn't accept w so $L(M_w) = \varnothing$ and thus $\langle M_w \rangle \notin \overline{E_{\mathsf{TM}}}$

One missing detail

What happens if the input to our T does not have the form $\langle M, w \rangle$?

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What happens if the input to our T does not have the form $\langle M, w \rangle$?

We said it outputs ε but that's actually a problem; why?

$\varepsilon \in \overline{E_{\mathsf{TM}}}$

We need to modify T:
T = "On input w,

1. If w isn't of the form $\langle M, w \rangle$, then output $\langle M' \rangle$ where $L(M') = \varnothing$
2. Otherwise, construct M_w = 'On input x,
   1. Run M on w. If M accepts, then *accept*; if M rejects, then *reject*'
3. Output $\langle M_w \rangle$"

Now strings that don't have the appropriate form for A_{TM} are mapped to something that's not in $\overline{E_{\mathsf{TM}}}$

Example

We showed that $A_{\mathsf{TM}} \leq E_{\mathsf{TM}}$ when we proved that E_{TM} is undecidable; show that $A_{\mathsf{TM}} \nleq_{\mathrm{m}} E_{\mathsf{TM}}$

How do we show this?

Example

We showed that $A_{\mathsf{TM}} \leq E_{\mathsf{TM}}$ when we proved that E_{TM} is undecidable; show that $A_{\mathsf{TM}} \not\leq_{\mathrm{m}} E_{\mathsf{TM}}$
How do we show this?

By contradiction. Assume that $A_{\mathsf{TM}} \leq_{\mathrm{m}} E_{\mathsf{TM}}$. We previously showed that E_{TM} is coRE so therefore A_{TM} is coRE. But this is a contradiction because we also proved that A_{TM} is *not* coRE

Languages that are neither RE nor coRE

So far, we've seen languages like A_{TM} that are RE but not coRE and languages like E_{TM} that are coRE but not RE

It's reasonable to ask if a language must be either RE or coRE. The answer is no

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The language EQ_{TM} is neither RE nor coRE

To prove this, we want to find two languages A and B such that $A \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$ and $B \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$ where A is not RE and B is not coRE

EQ_{TM} is not RE

We already showed $E_{\mathsf{TM}} \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$ and E_{TM} is not RE so EQ_{TM} is not RE

EQ_{TM} is not coRE

This one is a bit trickier. Let's mapping reduce A_{TM} to EQ_{TM}

How do we do this?

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How do we do this?

T = "On input $\langle M, w \rangle$,

1. Construct TM M_1 = 'On input x,
   1. If $x \neq w$, then *reject*
   2. Run M on w. If M accepts, then *accept*; if M rejects, then *reject*'
2. Construct TM M_2 = 'On input x,
   1. If $x = w$, then *accept*; otherwise *reject*'
3. Output $\langle M_1, M_2 \rangle$"

EQ_{TM} is not coRE

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If $\langle M, w\rangle \in A_{\mathsf{TM}}$, then M accepts w so $L(M_1) = \{w\}$. If $\langle M, w\rangle \notin A_{\mathsf{TM}}$, then M does not accept w so $L(M_1) = \varnothing$

Regardless of M, the language of M_2 is $L(M_2) = \{w\}$.

Thus $\langle M, w\rangle \in A_{\mathsf{TM}}$ iff $\langle M_1, M_2\rangle \in EQ_{\mathsf{TM}}$

Question 4

Is there a RE language A such that $EQ_{\mathsf{TM}} \leq_{\mathrm{m}} A$? Why or why not?

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No. EQ_{TM} is not RE, so any A such that $EQ_{\mathsf{TM}} \leq_{\mathrm{m}} A$ is also not RE

Question 5

Is there a coRE language B such that $B \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$? Why or why not?

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Yes. We showed $E_{\mathsf{TM}} \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$ and E_{TM} is coRE

Question 6

If C is a language and $EQ_{\mathsf{TM}} \leq_{\mathrm{m}} C$, what can we conclude about C?

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C is neither RE nor coRE

Question 7

True or false: If $D \leq E$, then $D \leq_{\mathrm{m}} E$.

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False. $A_{\mathsf{TM}} \leq E_{\mathsf{TM}}$ but $A_{\mathsf{TM}} \not\leq_{\mathrm{m}} E_{\mathsf{TM}}$

Question 8

Tricky! If $F \leq_{\mathrm{m}} \Sigma^*$, what can we conclude about F?

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$F = \Sigma^*$. Let f be the mapping. Then $w \in F \iff f(w) \in \Sigma^*$

For any language other than Σ^*, there's some string x not in the language but then $f(x) \notin \Sigma^*$; but every string is in Σ^*

Question 9

Tricky! If $\Sigma^* \leq_{\mathrm{m}} G$, what can we conclude about G?

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Tricky! If $\Sigma^* \leq_{\mathrm{m}} G$, what can we conclude about G?

We know $G \neq \varnothing$.

Since every string $w \in \Sigma^*$ needs to be mapped to an element of G, G cannot be empty

Updated table

Before today's lecture

Language	RE	coRE
A_{DFA}	✓	✓
E_{DFA}	✓	✓
EQ_{DFA}	✓	✓
A_{CFG}	✓	✓
E_{CFG}	✓	✓
EQ_{CFG}	✗	✓
DIAG	?	?
A_{TM}	✓	✗
HALT_{TM}	?	?
E_{TM}	✗	✓
EQ_{TM}	?	?
$\text{REGULAR}_{\text{TM}}$	?	?

Now

Language	RE	coRE
A_{DFA}	✓	✓
E_{DFA}	✓	✓
EQ_{DFA}	✓	✓
A_{CFG}	✓	✓
E_{CFG}	✓	✓
EQ_{CFG}	✗	✓
DIAG	?	?
A_{TM}	✓	✗
HALT_{TM}	?	✗
E_{TM}	✗	✓
EQ_{TM}	✗	✗
$\text{REGULAR}_{\text{TM}}$	?	?

$\textsc{Halt}_{\textsf{TM}}$ is RE

It's easy to show that $\textsc{Halt}_{\textsf{TM}}$ is RE

1. Construct a TM that recognizes $\textsc{Halt}_{\textsf{TM}}$
   H = "On input $\langle M, w\rangle$,
   1. Run M on w. If M halts, then *accept*"

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1. Construct a TM that recognizes $\textsc{Halt}_{\mathsf{TM}}$
   H = "On input $\langle M, w \rangle$,
   1. Run M on w. If M halts, then *accept*"
2. Mapping reduce $\textsc{Halt}_{\mathsf{TM}}$ to A_{TM}
   T = "On input $\langle M, w \rangle$,
   1. Construct TM M' = 'On input x,
      1. Run M on x. If M halts, then *accept*'
   2. Output $\langle M', w \rangle$"

Turning a Turing reduction into a mapping reduction

If the Turing reduction $A \leq B$ looks like:

Let R decide B and construct TM M to decide A:
M = "On input w,

1. Construct some instance w' of B
2. Run R on w' and if R accepts, then *accept*; otherwise *reject*"

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Note that R must be used exactly one time and M accepts iff R accepts

$\textsc{Regular}_{\textsf{TM}}$ is not coRE

We can turn our reduction $A_{\textsf{TM}} \leq \textsc{Regular}_{\textsf{TM}}$ into a mapping reduction $A_{\textsf{TM}} \leq_{\mathrm{m}} \textsc{Regular}_{\textsf{TM}}$

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T = "On input $\langle M, w \rangle$,

1. Construct TM M' = 'On input x,
   1. If $x = 0^n 1^n$ for some n, then *accept*
   2. Otherwise, run M on w and if M accepts, then *accept*; if M rejects, then *reject*'
2. Output $\langle M' \rangle$"

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$$\langle M, w \rangle \in A_{\textsf{TM}} \iff L(M') = \Sigma^* \iff L(M') \text{ is regular} \iff \langle M' \rangle \in \textsc{Regular}_{\textsf{TM}}$$

$A_{\textsf{TM}}$ is not coRE, so $\textsc{Regular}_{\textsf{TM}}$ is not coRE

$\text{REGULAR}_{\text{TM}}$ is not RE

We could reduce from E_{TM}, but it's simpler to reduce from $\overline{A_{\text{TM}}}$

T = "On input s,

1. If $s \neq \langle M, w \rangle$ for some TM M and input w, let M' be a TM such that $L(M') = \varnothing$
2. Otherwise, construct TM M' = 'On input x,
   1. If $x \neq 0^n 1^n$ for some n, then *reject*
   2. Run M on w and if M accepts, then *accept*; if M rejects, then *reject*'
3. Output $\langle M' \rangle$"

Three cases

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3. Output $\langle M'\rangle$"

Three cases

1. If $s \in \overline{A_{\mathsf{TM}}}$ but $s \neq \langle M, w\rangle$, then $L(M) = \varnothing$ and $\langle M'\rangle \in \textsc{Regular}_{\mathsf{TM}}$

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3. Output $\langle M'\rangle$"

Three cases

1. If $s \in \overline{A_{\mathsf{TM}}}$ but $s \neq \langle M, w\rangle$, then $L(M) = \varnothing$ and $\langle M'\rangle \in \textsc{Regular}_{\mathsf{TM}}$
2. If $s = \langle M, w\rangle \in \overline{A_{\mathsf{TM}}}$, then $w \notin L(M)$ so $L(M') = \varnothing$ and $\langle M'\rangle \in \textsc{Regular}_{\mathsf{TM}}$

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T = "On input s,

1. If $s \neq \langle M, w \rangle$ for some TM M and input w, let M' be a TM such that $L(M') = \emptyset$
2. Otherwise, construct TM M' = 'On input x,
   1. If $x \neq 0^n 1^n$ for some n, then *reject*
   2. Run M on w and if M accepts, then *accept*; if M rejects, then *reject*'
3. Output $\langle M' \rangle$"

Three cases

1. If $s \in \overline{A_{\text{TM}}}$ but $s \neq \langle M, w \rangle$, then $L(M) = \emptyset$ and $\langle M' \rangle \in \textsc{Regular}_{\text{TM}}$
2. If $s = \langle M, w \rangle \in \overline{A_{\text{TM}}}$, then $w \notin L(M)$ so $L(M') = \emptyset$ and $\langle M' \rangle \in \textsc{Regular}_{\text{TM}}$
3. If $s \notin \overline{A_{\text{TM}}}$, then $s = \langle M, w \rangle$ and $w \in L(M)$. In this case, $L(M') = \{0^n 1^n \mid n \geq 0\}$ so $\langle M' \rangle \notin \textsc{Regular}_{\text{TM}}$

Since $\overline{A_{\text{TM}}}$ is not RE, $\textsc{Regular}_{\text{TM}}$ is not RE

Updated table

Before today's lecture

Language	RE	coRE
A_{DFA}	✔	✔
E_{DFA}	✔	✔
EQ_{DFA}	✔	✔
A_{CFG}	✔	✔
E_{CFG}	✔	✔
EQ_{CFG}	✖	✔
DIAG	?	?
A_{TM}	✔	✖
$\text{HALT}_{\mathsf{TM}}$	?	?
E_{TM}	✖	✔
EQ_{TM}	?	?
$\text{REGULAR}_{\mathsf{TM}}$	?	?

Now

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A_{DFA}	✔	✔
E_{DFA}	✔	✔
EQ_{DFA}	✔	✔
A_{CFG}	✔	✔
E_{CFG}	✔	✔
EQ_{CFG}	✖	✔
DIAG	?	?
A_{TM}	✔	✖
$\text{HALT}_{\mathsf{TM}}$	✔	✖
E_{TM}	✖	✔
EQ_{TM}	✖	✖
$\text{REGULAR}_{\mathsf{TM}}$	✖	✖