CS 383

Lecture 19 - Diagonalization and undecidable languages

Stephen Checkoway

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Sizes of sets

Two sets X and Y have the same size if there is a bijection between them, $f: X \to Y$ What's a bijection?

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Recall $f: X \to Y$ is a bijection if

1 for all $a, b \in X$, f(a) = f(b) implies a = b (injective)

2 for all $y \in Y$, there exists $x \in X$ such that y = f(x) (surjective)

The natural numbers and the integers have the same size

$$f: \mathbb{Z} \to \mathbb{N}$$
$$f(x) = \begin{cases} 2x & \text{if } x \ge 0\\ -2x - 1 & \text{if } x < 0 \end{cases}$$

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- \vdots $-2 \mapsto 3$ $-1 \mapsto 1$ $0 \mapsto 0$ $1 \mapsto 2$ $2 \mapsto 4$
 - :

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The fundamental theorem of arithmetic tells us that every positive integer can be expressed uniquely as a product of prime powers

 $p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots$

where p_i are the primes in order (2, 3, 5, 7, etc.) and $n_i \in \mathbb{N}$ and finitely many n_i are nonzero

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Similarly, every positive rational number can be expressed uniquely as a product of prime powers

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where p_i are the primes in order and $n_i \in \mathbb{Z}$ and finitely many n_i are nonzero

Let $f:\mathbb{Z}\to\mathbb{N}$ be our bijection from before Define $g:\mathbb{Q}^+\to\mathbb{Z}^+$ by

$$g(p_1^{n_1}p_2^{n_2}p_3^{n_3}\cdots) = p_1^{f(n_1)}p_2^{f(n_2)}p_3^{f(n_3)}\cdots$$

Note that we're mapping the integer exponents to natural number exponents and the (infinitely many) 0 exponents remain 0 because f(0) = 0

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Finally, let's define our bijection $h:\mathbb{Q}\to\mathbb{Z}$

$$h(x) = \begin{cases} g(x) & \text{if } x > 0\\ 0 & \text{if } x = 0\\ -g(-x) & \text{if } x < 0 \end{cases}$$

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And just for fun, $f \circ h : \mathbb{Q} \to \mathbb{N}$ is a bijection

Countable

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Countably infinite sets include $\mathbb N,\,\mathbb Z,$ and $\mathbb Q$

Subsets of countable sets are countable (intuitively true but a hassle to prove without some additional math or an alternative, but equivalent definition of countability)

Each language is a countable set

Given an alphabet Σ , the language Σ^* is countably infinite. How do we show this?

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List the strings in lexicographic order to construct the mapping E.g., $f: \{0,1\}^* \to \mathbb{N}$ given by

 $\varepsilon \mapsto 0$ $0 \mapsto 1$ $1 \mapsto 2$ $00 \mapsto 3$ $01 \mapsto 4$ $10 \mapsto 5$ $11 \mapsto 6$ $000 \mapsto 7$:

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Every language $L \subseteq \Sigma^*$ is thus countable

Diagonalization: infinite sequences over $\{0, 1\}$

Theorem

The set S of all infinite sequences over $\{0,1\}$ is uncountable

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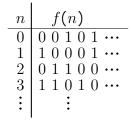
Theorem

The set S of all infinite sequences over $\{0,1\}$ is uncountable

Proof.

Assume S is countable so there's a bijection $f:\mathbb{N} \rightarrow S$

We can construct a new infinite sequence $\mathbf{b} = b_0, b_1, \dots$ that differs from every sequence in S.



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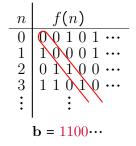
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In particular, b_i will differ from f(i) in position i

$$b_i = \begin{cases} 0 & \text{if the } i\text{th element of } f(i) \text{ is } 1\\ 1 & \text{if the } i\text{th element of } f(i) \text{ is } 0 \end{cases}$$



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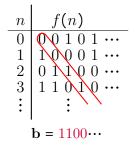
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Now $\mathbf{b} \in S$ but for all $i, f(i) \neq \mathbf{b}$ which is a contradiction so S must not be countable



There are a countable number of Turing machines

Consider any fixed binary representation of a TM

E.g., given

$$Q = \{1, 2, \dots, k\}$$

$$\Sigma = \{1, 2, \dots, m\}$$

$$\Gamma = \{1, 2, \dots, n\}$$

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{1, 2\}$$
where 1 = L and 2 = R

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$$

here's one possible representation

$$\begin{split} \langle \delta(q,a) \rangle &= 0^{r} 10^{b} 10^{d} & \text{where } \delta(q,a) = (r,b,d) \\ \langle \delta \rangle &= \langle \delta(1,1) \rangle \ 11 \ \langle \delta(1,2) \rangle \ 11 \cdots \ 11 \ \langle \delta(k,n) \rangle \\ \langle M \rangle &= 0^{k} \ 111 \ 0^{m} \ 111 \ 0^{n} \ 111 \ \langle \delta \rangle \ 111 \ 0^{q_{\text{accept}}} \ 111 \ 0^{q_{\text{reject}}} \end{split}$$

Thus $\langle M \rangle$ is an element of $\{0,1\}^*$

There are a countable number of Turing machines continued

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Now every binary string is a valid encoding of a TM, i.e.,

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For simplicity, for all $x \in \{0, 1\}^*$ such that x is not a valid encoding of a TM, define x to be a TM with $q_0 = q_{\text{reject}}$

Now every binary string is a valid encoding of a TM, i.e.,

 $\{0,1\}^* = \{\langle M \rangle \mid \langle M \rangle \text{ is is a TM}\}$

Since $\{0,1\}^*$ is countable, there are a countable number of Turing machines

There are an uncountable number of languages

Theorem

For every alphabet $\Sigma,$ the set of all languages over Σ is uncountable

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Proof.

We proved that $\boldsymbol{\Sigma}^*$ is countably infinite; let $f:\mathbb{N}\to\boldsymbol{\Sigma}^*$ be a bijection

For each language L over $\Sigma,$ define an infinite sequence \mathbf{b} = b_0, b_1, \ldots over $\{0,1\}$ where

$$b_i = \begin{cases} 0 & \text{if } f(i) \notin L \\ 1 & \text{if } f(i) \in L \end{cases}$$

 ${\bf b}$ is called the characteristic sequence of L

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Each characteristic sequence defines a language and each language has a unique characteristic sequence

We proved that there are uncountably many infinite binary sequences so there are uncountably many languages over Σ

A simple corollary

There are (uncountably many) languages that are not Turing-recognizable (and thus not decidable)

Theorem

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• If $\langle D \rangle \in \text{DIAG}$, then since D decides DIAG, D must accept $\langle D \rangle$ but then by definition of DIAG, $\langle D \rangle \notin \text{DIAG}$

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- If $\langle D \rangle \notin \text{DIAG}$, then since D decides DIAG, D must reject $\langle D \rangle$ but if D rejects $\langle D \rangle$, then by definition, $\langle D \rangle \in \text{DIAG}$

Either option leads to a contradiction so $\mathrm{DIAG}\xspace$ must not be decidable

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Replacing "reject" with "does not accept" in the proof shows that $\rm DIAG$ is not only not decidable, it's not even Turing-recognizable!

Acceptance problem for TMs

Theorem

The language $A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } w \in L(M) \}$ is undecidable

How should we approach problems like this?

Proving that a language is not decidable

To prove that a language A is undecidable,

- () Assume that A is decidable and let R be a TM that decides A
- **2** Select an undecidable language B
- **3** Construct a new TM D that decides B and that uses R as a subroutine
- Since B is undecidable but D is a decider, this is a contradiction and our assumption in step 1 must be wrong so A is undecidable

Steps 2 and 3 are the hard steps that require some cleverness

Proof that $A_{\rm TM}$ is undecidable. Assume that $A_{\rm TM}$ is decidable with decider R.

Let's build a TM D that decides DIAG.

Proof that A_{TM} is undecidable.

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- $D = "On input \langle M \rangle$,
 - $\textcircled{1} \operatorname{Run} R \text{ on } \langle M, \langle M \rangle \rangle$
 - **2** If R accepts, *reject*; otherwise *accept*."

We need to show that L(D) = DIAG and that D is a decider.

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By assumption, R is a decider so it halts on $\langle M,\langle M\rangle\rangle$ and thus D halts on all input so it is a decider

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If $\langle M \rangle \in \text{DIAG}$, then $\langle M \rangle \notin L(M)$ so R rejects and D accepts so $\langle M \rangle \in L(D)$.

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Thus D decides DIAG. This is a contradiction so A_{TM} must not be decidable.

Halting problem for TMs

Theorem

The language $H_{ALT_{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts when run on } w \}$ is undecidable

Assume that $HALT_{TM}$ is decided by TM H. How do we use H to construct a decider D for A_{TM} ?

Proof.

Assume H is a decider for HALT_{TM} and build a decider D for A_{TM} . D = "On input $\langle M, w \rangle$,

- **1** Run H on $\langle M, w \rangle$ and if H rejects, *reject*.
- **2** Run M on w and if M accepts, *accept*; otherwise *reject*."

D is a decider because if M loops on w, then H and D will reject. Otherwise, M will halt on w so D will halt.

If $w \in L(M)$, then M halts on w so H will accept and then D will accept.

If $w \notin L(M)$, then there are two options. If M loops on w, then H and thus D will reject. If M rejects w, then H will accept but D will reject.

Co-Turing-recognizable (CoRE)

A language L is co-Turing-recognizable (coRE) if \overline{L} is Turing-recognizable (RE)

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A language L is co-Turing-recognizable (coRE) if \overline{L} is Turing-recognizable (RE)

Theorem

A language L is decidable $\iff L$ is RE and L is coRE

To prove this, we need to prove three things

- 1) If L is decidable, then L is RE
- **2** If L is decidable, then L is coRE
- **3** If L is RE and coRE, then L is decidable

Parts 1 and 2 together show the \implies direction and part 3 shows the \iff direction

Proof.

 \implies :

If L is decidable, then there is some decider M such that L(M) = L. Thus L is RE.

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;=⇒

If L is RE, then there is some TM M_1 that recognizes it If L is coRE, then there is some TM M_2 that recognizes \overline{L}

Build M = "On input w,

1 Run M_1 and M_2 on w simultaneously (e.g., with 2 tapes)

2 If M_1 accepts, *accept*. If M_2 accepts, *reject*."

One of M_1 or M_2 must accept, so M will halt on any input and thus decides L.

$A_{\rm TM}$ is RE but not coRE

Theorem

 A_{TM} is RE but not coRE

Proof.

Since A_{TM} is not decidable, if we show that it is RE, then it can't be coRE because then it would be decidable.

We can build R to recognize A_{TM} as follows. $R = \text{``On input } \langle M, w \rangle$,

 $\textcircled{1} \mathsf{Run} \ M \ \mathsf{on} \ w.$

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2 If M accepts, *accept*; if M rejects, *reject*."

Note that if M loops on w, then R will loop, but this is okay because R just needs to recognize $A_{\rm TM},$ not decide it

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- **3** The input isn't a valid encoding of $\langle M, w \rangle$. R will reject before step 1.

There are three cases

- (1) (M, w) ∈ A_{TM} . M will accept w so R will accept.
- ② $\langle M, w \rangle \notin A_{\mathsf{TM}}$. *M* will either loop on *w* or reject and *R* will do the same.
- **3** The input isn't a valid encoding of $\langle M, w \rangle$. R will reject before step 1.

Thus $L(R) = A_{\text{TM}}$ so A_{TM} is RE.

Theorem

The language $E_{TM} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \}$ is coRE.

To prove this, we need only give a TM that recognizes $\overline{E_{\mathsf{TM}}}$

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Proof.

Let R = "On input w,

- 1 If $w \neq \langle M \rangle$ for some TM M, accept.
- **2** For n = 0 up to ∞
- **3** For each string $w \in \Sigma^*$ of length at most n
- 4 Simulate M on w for at most n steps.
- **5** If M accepts w, accept."

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If $L(M) \neq \emptyset$, then there is some w that M will accept so R will accept $\langle M \rangle$.

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- **4** Simulate M on w for at most n steps.
- **5** If M accepts w, accept."

If $L(M) \neq \emptyset$, then there is some w that M will accept so R will accept $\langle M \rangle$.

If $L(M) = \emptyset$, then M will never accept so R will loop on $\langle M \rangle$.

Thus $L(R) = \overline{E_{TM}}$ so E_{TM} is coRE.

Emptiness problem for TMs is undecidable

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Emptiness problem for TMs is undecidable

Theorem The language E_{TM} is undecidable. Corollary

The language E_{TM} is not RE.

Proof of the corollary.

Since $E_{\rm TM}$ is coRE, if it were RE, then it would be decidable, contradicting the theorem.

Proof idea for showing E_{TM} is undecidable

- Assume *E* decides *E*_{TM}
- Build a decider for A_{TM} using E
- Along the way, we're going to construct an entirely new TM M_w and we're going to run E on $\langle M_w \rangle$

We'll use the idea of constructing new TMs in a bunch of different proofs

Proof.

Assume that E decides E_{TM} . Build D to decide A_{TM} .

D ="On input $\langle M, w \rangle$,

1 Construct a new TM M_w = 'On any input x,

1 Replace x on the tape with w and run M on w.

- **2** If M accepts, *accept*; if M rejects, *reject*.'
- **2** Run E on $\langle M_w \rangle$.

3 If E accepts, reject; otherwise accept."

Note that M_w is never run. It is only constructed so that $\langle M_w \rangle$ can be given as input to decider E.

Proof.

Assume that E decides E_{TM} . Build D to decide A_{TM} .

D ="On input $\langle M, w \rangle$,

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If $w \in L(M)$, then $L(M_w) = \Sigma^* \neq \emptyset$ so E rejects and D accepts.

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If $w \notin L(M)$, then $L(M_w) = \emptyset$ so E accepts and D rejects. Thus $L(D) = E_{\mathsf{TM}}$.

Proof.

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1 Replace x on the tape with w and run M on w.

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Constructing M_w can't loop and E is a decider so D is a decider.