CS 383

Lecture 19 – Diagonalization and undecidable languages

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Sizes of sets

Two sets X and Y have the same size if there is a bijection between them, $f : X \to Y$

What's a bijection?

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Recall $f : X \to Y$ is a bijection if

1. for all $a, b \in X$, $f(a) = f(b)$ implies $a = b$ (injective)
2. for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$ (surjective)

Example

The natural numbers and the integers have the same size

$$f : \mathbb{Z} \to \mathbb{N}$$

$$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x - 1 & \text{if } x < 0 \end{cases}$$

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$$\begin{aligned} &\vdots \\ -2 &\mapsto 3 \\ -1 &\mapsto 1 \\ 0 &\mapsto 0 \\ 1 &\mapsto 2 \\ 2 &\mapsto 4 \\ &\vdots \end{aligned}$$

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The fundamental theorem of arithmetic tells us that every positive integer can be expressed uniquely as a product of prime powers

$$p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots$$

where p_i are the primes in order (2, 3, 5, 7, etc.) and
$n_i \in \mathbb{N}$ and finitely many n_i are nonzero

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where p_i are the primes in order (2, 3, 5, 7, etc.) and
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Similarly, every positive rational number can be expressed uniquely as a product of prime powers

$$p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots$$

where p_i are the primes in order and
$n_i \in \mathbb{Z}$ and finitely many n_i are nonzero

Example continued

Let $f : \mathbb{Z} \to \mathbb{N}$ be our bijection from before
Define $g : \mathbb{Q}^+ \to \mathbb{Z}^+$ by

$$g(p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots) = p_1^{f(n_1)} p_2^{f(n_2)} p_3^{f(n_3)} \cdots$$

Note that we're mapping the integer exponents to natural number exponents and the (infinitely many) 0 exponents remain 0 because $f(0) = 0$

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Since f is a bijection, g is a bijection (this isn't hard to show)

Finally, let's define our bijection $h : \mathbb{Q} \to \mathbb{Z}$

$$h(x) = \begin{cases} g(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -g(-x) & \text{if } x < 0 \end{cases}$$

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And just for fun, $f \circ h : \mathbb{Q} \to \mathbb{N}$ is a bijection

Countable

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Subsets of countable sets are countable (intuitively true but a hassle to prove without some additional math or an alternative, but equivalent definition of countability)

Each language is a countable set

Given an alphabet Σ, the language Σ^* is countably infinite. How do we show this?

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List the strings in lexicographic order to construct the mapping
E.g., $f : \{0, 1\}^* \to \mathbb{N}$ given by

$$\begin{aligned} \varepsilon &\mapsto 0 \\ 0 &\mapsto 1 \\ 1 &\mapsto 2 \\ 00 &\mapsto 3 \\ 01 &\mapsto 4 \\ 10 &\mapsto 5 \\ 11 &\mapsto 6 \\ 000 &\mapsto 7 \\ &\vdots \end{aligned}$$

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Given an alphabet Σ, the language Σ^* is countably infinite. How do we show this?

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Every language $L \subseteq \Sigma^*$ is thus countable

Diagonalization: infinite sequences over $\{0, 1\}$

Theorem

The set S of all infinite sequences over $\{0, 1\}$ is uncountable

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Proof.

Assume S is countable so there's a bijection $f : \mathbb{N} \to S$

We can construct a new infinite sequence $\mathbf{b} = b_0, b_1, \ldots$ that differs from every sequence in S.

n	$f(n)$
0	0 0 1 0 1 $\cdots$
1	1 0 0 0 1 $\cdots$
2	0 1 1 0 0 $\cdots$
3	1 1 0 1 0 $\cdots$
$\vdots$	$\vdots$

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In particular, b_i will differ from $f(i)$ in position i

$$b_i = \begin{cases} 0 & \text{if the } i\text{th element of } f(i) \text{ is } 1 \\ 1 & \text{if the } i\text{th element of } f(i) \text{ is } 0 \end{cases}$$

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$\mathbf{b} = 1100\cdots$

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Now $\mathbf{b} \in S$ but for all i, $f(i) \neq \mathbf{b}$ which is a contradiction so S must not be countable □

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$\mathbf{b} = 1100\cdots$

There are a countable number of Turing machines

Consider any fixed binary representation of a TM

E.g., given

$$Q = \{1, 2, \ldots, k\}$$
$$\Sigma = \{1, 2, \ldots, m\}$$
$$\Gamma = \{1, 2, \ldots, n\}$$
$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{1, 2\}$$ where $1 = \mathrm{L}$ and $2 = \mathrm{R}$
$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$$

here's one possible representation

$$\langle \delta(q, a) \rangle = 0^r 1 0^b 1 0^d$$ where $\delta(q, a) = (r, b, d)$
$$\langle \delta \rangle = \langle \delta(1,1) \rangle \ 11 \ \langle \delta(1,2) \rangle \ 11 \cdots 11 \ \langle \delta(k,n) \rangle$$
$$\langle M \rangle = 0^k \ 111 \ 0^m \ 111 \ 0^n \ 111 \ \langle \delta \rangle \ 111 \ 0^{q_{\text{accept}}} \ 111 \ 0^{q_{\text{reject}}}$$

Thus $\langle M \rangle$ is an element of $\{0, 1\}^*$

There are a countable number of Turing machines continued

For simplicity, for all $x \in \{0,1\}^*$ such that x is not a valid encoding of a TM, define x to be a TM with $q_0 = q_{\text{reject}}$

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Now every binary string is a valid encoding of a TM, i.e.,

$$\{0, 1\}^* = \{\langle M \rangle \mid \langle M \rangle \text{ is is a TM}\}$$

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For simplicity, for all $x \in \{0,1\}^*$ such that x is not a valid encoding of a TM, define x to be a TM with $q_0 = q_{\text{reject}}$

Now every binary string is a valid encoding of a TM, i.e.,

$$\{0,1\}^* = \{\langle M \rangle \mid \langle M \rangle \text{ is is a TM}\}$$

Since $\{0,1\}^*$ is countable, there are a countable number of Turing machines

There are an uncountable number of languages

Theorem

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Proof.

We proved that Σ^* is countably infinite; let $f : \mathbb{N} \to \Sigma^*$ be a bijection

For each language L over Σ, define an infinite sequence $\mathbf{b} = b_0, b_1, \ldots$ over $\{0, 1\}$ where

$$b_i = \begin{cases} 0 & \text{if } f(i) \notin L \\ 1 & \text{if } f(i) \in L \end{cases}$$

$\mathbf{b}$ is called the characteristic sequence of L

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$\mathbf{b}$ is called the characteristic sequence of L

Each characteristic sequence defines a language and each language has a unique characteristic sequence

We proved that there are uncountably many infinite binary sequences so there are uncountably many languages over Σ □

A simple corollary

There are (uncountably many) languages that are not Turing-recognizable (and thus not decidable)

An explicit undecidable language

Theorem
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Assume that D is a TM that decides $\textsc{Diag}$

Is $\langle D \rangle \in \textsc{Diag}$?

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Two options

- If $\langle D \rangle \in \textsc{Diag}$, then since D decides $\textsc{Diag}$, D must accept $\langle D \rangle$ but then by definition of $\textsc{Diag}$, $\langle D \rangle \notin \textsc{Diag}$

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- If $\langle D \rangle \notin \textsc{Diag}$, then since D decides $\textsc{Diag}$, D must reject $\langle D \rangle$ but if D rejects $\langle D \rangle$, then by definition, $\langle D \rangle \in \textsc{Diag}$

Either option leads to a contradiction so $\textsc{Diag}$ must not be decidable □

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Either option leads to a contradiction so $\textsc{Diag}$ must not be decidable $\square$

Replacing "reject" with "does not accept" in the proof shows that $\textsc{Diag}$ is not only not decidable, it's not even Turing-recognizable!

Acceptance problem for TMs

Theorem

The language $A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } w \in L(M)\}$ *is undecidable*

How should we approach problems like this?

Proving that a language is not decidable

To prove that a language A is undecidable,

1. Assume that A is decidable and let R be a TM that decides A
2. Select an undecidable language B
3. Construct a new TM D that decides B and that uses R as a subroutine
4. Since B is undecidable but D is a decider, this is a contradiction and our assumption in step 1 must be wrong so A is undecidable

Steps 2 and 3 are the hard steps that require some cleverness

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Assume that A_{TM} is decidable with decider R.

Let's build a TM D that decides $\textsc{Diag}$.

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Let's build a TM D that decides $\textsc{Diag}$.

D = "On input $\langle M \rangle$,

1. Run R on $\langle M, \langle M \rangle \rangle$
2. If R accepts, *reject*; otherwise *accept*."

We need to show that $L(D) = \textsc{Diag}$ and that D is a decider.

Proof

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Assume that A_{TM} is decidable with decider R.

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We need to show that $L(D) = \textsc{Diag}$ and that D is a decider.

By assumption, R is a decider so it halts on $\langle M, \langle M \rangle \rangle$ and thus D halts on all input so it is a decider

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If $\langle M \rangle \in \textsc{Diag}$, then $\langle M \rangle \notin L(M)$ so R rejects and D accepts so $\langle M \rangle \in L(D)$.

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Proof that A_{TM} is undecidable.

Assume that A_{TM} is decidable with decider R.

Let's build a TM D that decides $\textsc{Diag}$.

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1. Run R on $\langle M, \langle M \rangle \rangle$
2. If R accepts, *reject*; otherwise *accept*."

We need to show that $L(D) = \textsc{Diag}$ and that D is a decider.

By assumption, R is a decider so it halts on $\langle M, \langle M \rangle \rangle$ and thus D halts on all input so it is a decider

If $\langle M \rangle \in \textsc{Diag}$, then $\langle M \rangle \notin L(M)$ so R rejects and D accepts so $\langle M \rangle \in L(D)$.

If $\langle M \rangle \notin \textsc{Diag}$, then $\langle M \rangle \in L(M)$ so R accepts and D rejects so $\langle M \rangle \notin L(D)$.

Thus D decides $\textsc{Diag}$. This is a contradiction so A_{TM} must not be decidable. $\square$

Halting problem for TMs

Theorem

The language $\textsc{Halt}_{\mathsf{TM}} = \{\langle M, w\rangle \mid M$ *is a TM and* M *halts when run on* $w\}$ *is undecidable*

Assume that $\textsc{Halt}_{\mathsf{TM}}$ is decided by TM H. How do we use H to construct a decider D for A_{TM}?

Proof

Proof.

Assume H is a decider for $\text{HALT}_{\mathsf{TM}}$ and build a decider D for A_{TM}.

D = "On input $\langle M, w \rangle$,

1. Run H on $\langle M, w \rangle$ and if H rejects, *reject*.
2. Run M on w and if M accepts, *accept*; otherwise *reject*."

D is a decider because if M loops on w, then H and D will reject. Otherwise, M will halt on w so D will halt.

If $w \in L(M)$, then M halts on w so H will accept and then D will accept.

If $w \notin L(M)$, then there are two options. If M loops on w, then H and thus D will reject. If M rejects w, then H will accept but D will reject. $\square$

Co-Turing-recognizable (CoRE)

A language L is co-Turing-recognizable (coRE) if $\overline{L}$ is Turing-recognizable (RE)

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A language L is co-Turing-recognizable (coRE) if $\overline{L}$ is Turing-recognizable (RE)

Theorem

A language L is decidable $\iff$ L is RE and L is coRE

To prove this, we need to prove three things

1. If L is decidable, then L is RE
2. If L is decidable, then L is coRE
3. If L is RE and coRE, then L is decidable

Parts 1 and 2 together show the $\implies$ direction and part 3 shows the $\impliedby$ direction

Proof

Proof.

$\Longrightarrow$:

If L is decidable, then there is some decider M such that $L(M) = L$. Thus L is RE.

Proof

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$\implies$:

If L is decidable, then there is some decider M such that $L(M) = L$. Thus L is RE.

By swapping the accept and reject states of M, we get a new decider M' that decides $\overline{L}$. Thus L is coRE.

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By swapping the accept and reject states of M, we get a new decider M' that decides $\overline{L}$. Thus L is coRE.

$\impliedby$:

If L is RE, then there is some TM M_1 that recognizes it
If L is coRE, then there is some TM M_2 that recognizes $\overline{L}$

Build M = "On input w,

1. Run M_1 and M_2 on w simultaneously (e.g., with 2 tapes)
2. If M_1 accepts, *accept*. If M_2 accepts, *reject*."

One of M_1 or M_2 must accept, so M will halt on any input and thus decides L. □

A_{TM} is RE but not coRE

Theorem

A_{TM} is RE but not coRE

Proof.

Since A_{TM} is not decidable, if we show that it is RE, then it can't be coRE because then it would be decidable.

We can build R to recognize A_{TM} as follows.
R = "On input $\langle M, w \rangle$,

1. Run M on w.
2. If M accepts, *accept*; if M rejects, *reject*."

A_{TM} is RE but not coRE

Theorem

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Proof.

Since A_{TM} is not decidable, if we show that it is RE, then it can't be coRE because then it would be decidable.

We can build R to recognize A_{TM} as follows.

R = "On input $\langle M, w \rangle$,

1. Run M on w.
2. If M accepts, *accept*; if M rejects, *reject*."

Note that if M loops on w, then R will loop, but this is okay because R just needs to recognize A_{TM}, not decide it

Proof continued

There are three cases

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There are three cases

1. $\langle M, w\rangle \in A_{\mathsf{TM}}$. M will accept w so R will accept.
2. $\langle M, w\rangle \notin A_{\mathsf{TM}}$. M will either loop on w or reject and R will do the same.
3. The input isn't a valid encoding of $\langle M, w\rangle$. R will reject before step 1.

Proof continued

There are three cases

1. $\langle M, w\rangle \in A_{\mathsf{TM}}$. M will accept w so R will accept.
2. $\langle M, w\rangle \notin A_{\mathsf{TM}}$. M will either loop on w or reject and R will do the same.
3. The input isn't a valid encoding of $\langle M, w\rangle$. R will reject before step 1.

Thus $L(R) = A_{\mathsf{TM}}$ so A_{TM} is RE. □

Emptiness problem for TMs

Theorem

The language $E_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$ *is coRE.*

To prove this, we need only give a TM that recognizes $\overline{E_{\mathsf{TM}}}$

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Proof.

Let R = "On input w,

1. If $w \neq \langle M \rangle$ for some TM M, *accept*.
2. For $n = 0$ up to ∞
3. For each string $w \in \Sigma^*$ of length at most n
4. Simulate M on w for at most n steps.
5. If M accepts w, *accept*."

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If $L(M) \neq \emptyset$, then there is some w that M will accept so R will accept $\langle M\rangle$.

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3. For each string $w \in \Sigma^*$ of length at most n
4. Simulate M on w for at most n steps.
5. If M accepts w, *accept*."

If $L(M) \neq \varnothing$, then there is some w that M will accept so R will accept $\langle M \rangle$.

If $L(M) = \varnothing$, then M will never accept so R will loop on $\langle M \rangle$.

Thus $L(R) = \overline{E_{\mathsf{TM}}}$ so E_{TM} is coRE. □

Emptiness problem for TMs is undecidable

Theorem

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Proof of the corollary.

Since E_{TM} is coRE, if it were RE, then it would be decidable, contradicting the theorem. □

Proof idea for showing E_{TM} is undecidable

- Assume E decides E_{TM}
- Build a decider for A_{TM} using E
- Along the way, we're going to construct an entirely new TM M_w and we're going to run E on $\langle M_w \rangle$

We'll use the idea of constructing new TMs in a bunch of different proofs

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Assume that E decides E_{TM}. Build D to decide A_{TM}.
D = "On input $\langle M, w\rangle$,

1. Construct a new TM M_w = 'On any input x,
   1. Replace x on the tape with w and run M on w.
   2. If M accepts, *accept*; if M rejects, *reject*.'
2. Run E on $\langle M_w\rangle$.
3. If E accepts, *reject*; otherwise *accept*."

Note that M_w is never run. It is only constructed so that $\langle M_w\rangle$ can be given as input to decider E.

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If $w \in L(M)$, then $L(M_w) = \Sigma^* \neq \emptyset$ so E rejects and D accepts.

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If $w \notin L(M)$, then $L(M_w) = \varnothing$ so E accepts and D rejects. Thus $L(D) = E_{\mathsf{TM}}$.

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If $w \in L(M)$, then $L(M_w) = \Sigma^* \neq \varnothing$ so E rejects and D accepts.

If $w \notin L(M)$, then $L(M_w) = \varnothing$ so E accepts and D rejects. Thus $L(D) = E_{\text{TM}}$.

Constructing M_w can't loop and E is a decider so D is a decider. □