CS 383

Lecture 14 – Non-context-free languages

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Fall 2023

Review of "pumpable" languages

Recall we call a language L pumpable with pumping length p if for all $w \in L$ with $|w| \geq p$, there exist strings $x, y, z \in \Sigma^*$ with $w = xyz$ such that

1. for all $i \geq 0$, $xy^i z \in L$;
2. $|y| > 0$; and
3. $|xy| \leq p$

Then we proved that regular languages are pumpable

This let us prove a language was not regular by showing it isn't pumpable

CF-pumpability

A language L is CF-pumpable with pumping length p if for all $w \in L$ with $|w| \geq p$, there exist strings $u, v, x, y, z \in \Sigma^*$ such that

1. for all $i \geq 0$, $uv^ixy^iz \in L$;
2. $|vy| > 0$; and
3. $|vxy| \leq p$

Rather than dividing the string into 3 pieces, we're dividing it into 5

Two of the pieces (v and y) are pumped together

Condition 2 tells us that at least one of v or y must not be ε

Example pumpable language

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Let $u = s$

$v = c$

$x = \#$

$y = c$

$z = s^{\mathcal{R}}$

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$$\begin{aligned} \text{Let } u &= s \\ v &= c \\ x &= \# \\ y &= c \\ z &= s^{\mathcal{R}} \end{aligned}$$

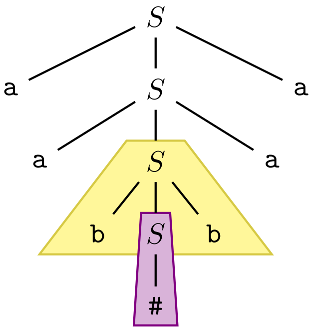
1. for any $i \geq 0$, $uv^ixy^iz = sc^i\#c^is^{\mathcal{R}} = (sc^i)\#(sc^i)^{\mathcal{R}} \in L$
2. $|vy| = |cc| = 2 > 0$
3. $|vxy| = |c\#c| = 3 \leq p$

Parse trees

CFG for $A = \{w\#w^{\mathcal{R}} \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$: $\quad S \to \mathtt{a}S\mathtt{a} \mid \mathtt{b}S\mathtt{b} \mid \#$

Consider a parse tree for $w = \mathtt{aab\#baa}$

$i = 1$:



$u = \mathtt{aa}$, $v = \mathtt{b}$, $x = \#$, $y = \mathtt{b}$, $z = \mathtt{aa}$

- Pumping down replaces the yellow trapezoid with the violet trapezoid
- Pumping up replaces the violet trapezoid with the yellow trapezoid

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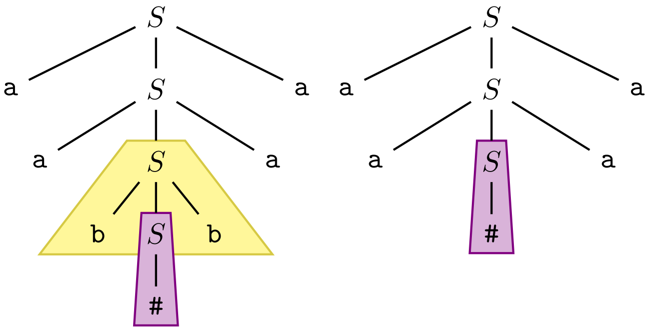

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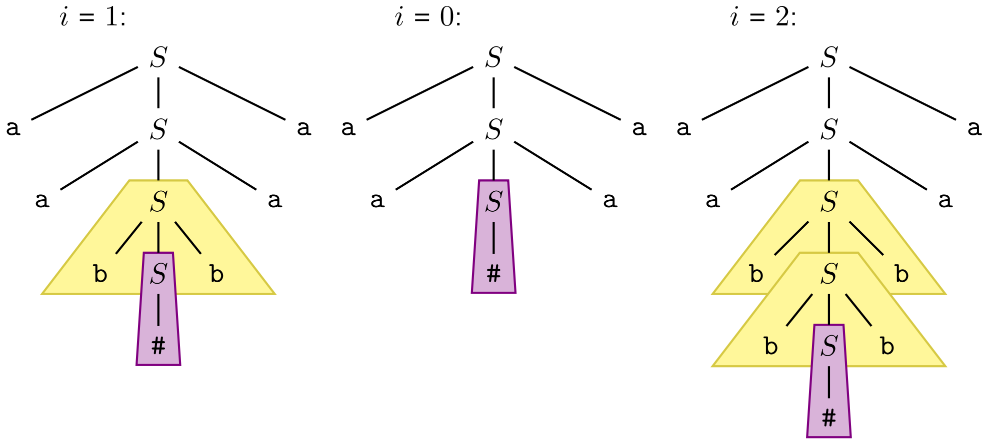

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CFLs are CF-pumpable

Theorem (Pumping lemma for context-free languages)

Context-free languages are CF-pumpable

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Proof idea.

Consider a CFG $G = (V, \Sigma, R, S)$ in CNF

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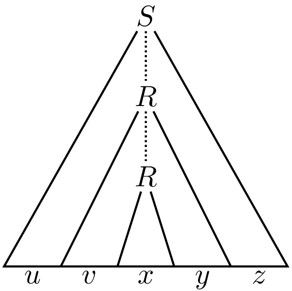
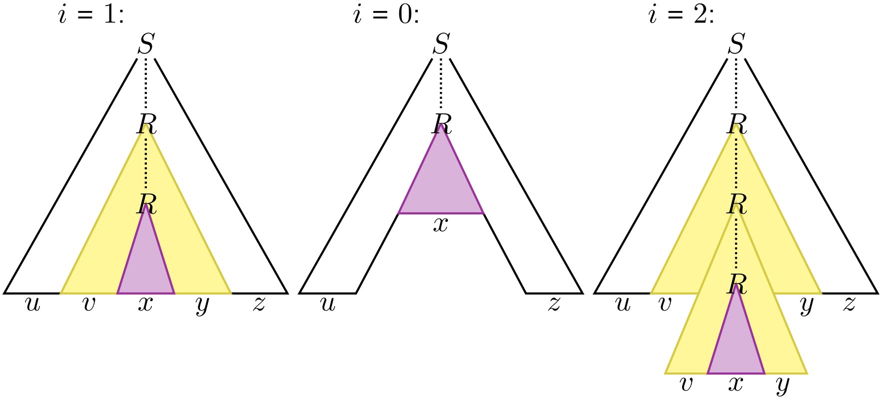
Set p large enough that any string of length at least p repeats some variable in its derivation (it turns out $p = 2^{|V|} + 1$ works)

This repeated variable, call it R, will play the same role as the repeated state did in proving that regular languages are pumpable

Note that this means $R \overset{*}{\Rightarrow} vxy$ and $R \overset{*}{\Rightarrow} x$



Condition 1: $\forall i \geq 0.\, uv^i xy^i z \in L$



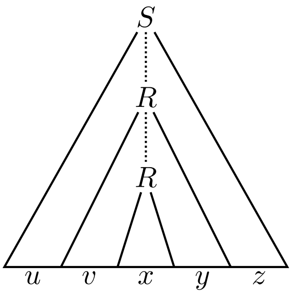
- Pumping down replaces the yellow triangle with the violet triangle
- Pumping up replaces the violet triangle with the yellow triangle
- We can pump up arbitrarily by repeating this process

Thus we've satisfied the first condition:

1. for all $i \geq 0$, $uv^i xy^i z \in L$

Condition 2: $|vy| > 0$

To see that at least one of v or y is not ε, let's look at $R \overset{*}{\Rightarrow} vRy$



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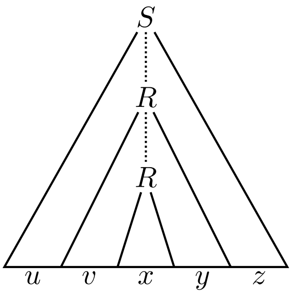
To see that at least one of v or y is not ε, let's look at $R \stackrel{*}{\Rightarrow} vRy$

Since G is in CNF, we must have $R \Rightarrow AB \stackrel{*}{\Rightarrow} vRy$ for some variables A and B

Two cases:



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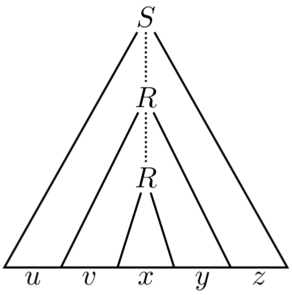
Two cases:

- $A \overset{*}{\Rightarrow} vRs$ and $B \overset{*}{\Rightarrow} t$ where $st = y$
  t (and thus y) cannot be ε because G is in CNF



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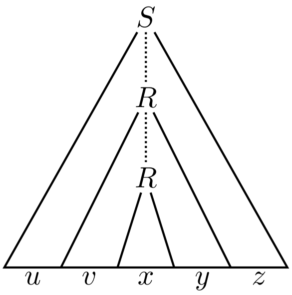

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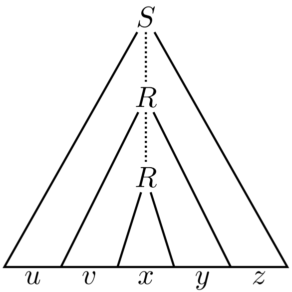

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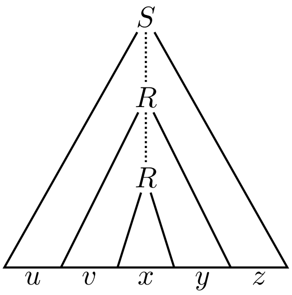
  s (and thus v) cannot be ε because G is in CNF

In either case, we've satisfied the second condition:

2. $|vy| > 0$

Condition 3: $|vxy| \leq p$

For strings with length at least $p = 2^{|V|} + 1$ we said there had to be a repeated variable



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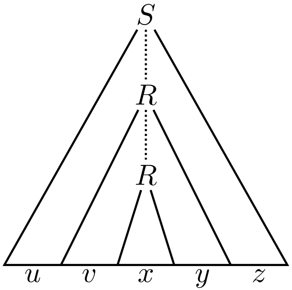
Looking at all parse trees of height at least $|V| + 1$, there must be a repeated variable (pigeonhole principle), so pick one of those in the lowest $|V| + 1$ for R that derives vxy



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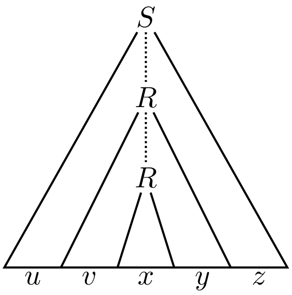
Now since R is at distance at most $|V|+1$ from the leaves, we must have $|vxy| \leq 2^{|V|} \leq p$

(A perfect binary tree of height h has 2^h leaves, but the last level of interior nodes in a parse tree for a grammar in CNF have a single child each)



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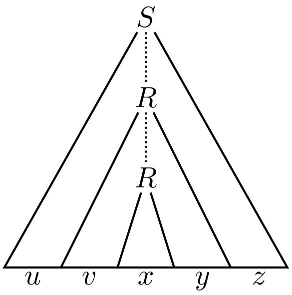
Therefore, we've satisfied the final condition:

3. $|vxy| \le p$



Showing that a language is not context-free

We can prove that a language is not context-free by showing that it violates the pumping lemma for context-free languages

Steps:

1. Assume the language, L, is context-free with some unspecified pumping length p
2. Pick string $w \in L$ such that $|w| \geq p$
3. Consider every division of w into $uvxyz = w$ such that $|vy| > 0$, and $|vxy| \leq p$
4. For each possible division, show that for some i, $uv^ixy^iz \notin L$

Example

$B = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$ is not context-free

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- At least one of v or y contains two distinct symbols. Then uv^2xy^2z contains symbols out of order so $uv^2xy^2z \notin B$
- Both v and y contain the same symbol ($v = \mathtt{a}^m$, $y = \mathtt{a}^n$; $v = \mathtt{b}^m$, $y = \mathtt{b}^n$; or $v = \mathtt{c}^m$, $y = \mathtt{c}^n$). Then uxz doesn't have the same number of as, bs, and cs, so $uv^0xy^0z \notin B$

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- v and y contain different symbols, but only a single type each ($v = \mathtt{a}^m$, $y = \mathtt{b}^n$; $v = \mathtt{a}^m$, $y = \mathtt{c}^n$; or $v = \mathtt{b}^m$, $y = \mathtt{c}^n$). Again, uxz doesn't have the same number of as, bs, and cs so $uv^0xy^0z \notin B$

Using closure properties

Using the pumping lemma for CFLs is a *pain*

We can prove that

$$C = \{w \mid w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \text{ and } w \text{ has the same number of } \mathtt{a}\text{s}, \mathtt{b}\text{s}, \text{ and } \mathtt{c}\text{s}\}$$

is not context-free by intersecting it with a regular language

What language should we choose?

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Intersect with $\underline{\mathtt{a}^*\mathtt{b}^*\mathtt{c}^*}$:

$$C \cap \underline{\mathtt{a}^*\mathtt{b}^*\mathtt{c}^*} = B$$

Since context-free languages are closed under intersection with a regular language, if C were context-free, then B would be context-free.

This is a contradiction so C is not context-free.

Another example

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- If v or y contains $\mathtt{b}$, then pumping down gives a string with too few $\mathtt{b}$s

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- If v or y contains b, then pumping down gives a string with too few bs
- If x doesn't contain a b, then $vxy = \mathtt{a}^m$ is in the first, second, or third run of as, for some m. In any case, pumping down gives a string with as in the wrong ratio

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- If x doesn't contain a b, then $vxy = \mathtt{a}^m$ is in the first, second, or third run of as, for some m. In any case, pumping down gives a string with as in the wrong ratio
- If x contains a b, then either $v = \mathtt{a}^m$ is in the first run of as and $y = \mathtt{a}^n$ is in the second, or v is in the second and y is in the third. In either case, pumping down gives a string with as in the wrong ratio

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Proofs using the pumping lemma always devolve to examining a bunch of cases

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- Use the fact that $|vxy| \leq p$ to constrain the cases; e.g., if you need some as followed by some bs followed by some cs, try to have at least p of each so that vxy cannot come from all three

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- Use the fact that $|vxy| \leq p$ to constrain the cases; e.g., if you need some as followed by some bs followed by some cs, try to have at least p of each so that vxy cannot come from all three
- Try to cover as many similar cases at once as possible; e.g., if several cases are analogous, try to address them in one argument

Intersection of CFLs

We know that the intersection of a CFL and a regular language is context-free

Is the intersection of two CFLs necessarily context-free?

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What are two context-free languages whose intersection is not context-free?

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$$\begin{aligned} E &= \{\mathtt{a}^m\mathtt{b}^m\mathtt{c}^n \mid m, n \geq 0\} \\ F &= \{\mathtt{a}^m\mathtt{b}^n\mathtt{c}^n \mid m, n \geq 0\} \\ E \cap F &= \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\} \end{aligned}$$